Abstract: In developed countries, car use is one of the most significant contributors to air pollution. It is also a notable fact that larger, heavier cars consume more fuel and hence contribute more to pollution. This observation has lead to policy proposals to structure taxation to encourage the use of smaller, lighter cars. A model of vertical product differentiation is used to explain why different types of car are chosen. The correct policy response to the externality problem is then investigated using this model. It is shown that there are reasons why the standard policy response may be flawed and that it may even be optimal to subsidize large cars at the expense of small. A comparison of policies reveals that the relative merits of differentiated annual taxes and differentiated fuel taxes are dependent on the degree of income inequality.

Acknowledgements: Thanks are due to John Conley, Peter Hammond, Michael Keen, David de Meza, Stephen Smith and participants at the 1999 Warwick Conference on Theoretical Environmental Economics.

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1. Introduction
When taking into consideration the necessity of travelling and communication in daily life, the significance of transportation cannot be denied. Expanded cities and developments in the nature of social activities have brought about an increase in the demand for transport. In turn, this increase in demand has lead to growing concern over transport-related environmental problems. Policies aimed at switching the demand for transport from less environmentally-friendly modes to more environmentally-friendly modes have been implemented. Among the policies designed to cope with the air pollution associated with the use of cars, the use of tax instruments has had much prominence. When applied to cars, both fuel duties and ownership taxes have been considered. This paper investigates the design of tax incentives, taking into account the factors involved in the choice of car type.

Transport is one of the main sources of air pollution due to the fuel used (Department of Transport (DOT) (1990, 1996)). Rapidly increasing car ownership is exacerbating the current situation and the effects of the pollutants emitted by transport on human health and the environment are widely recognized. Unless measures aimed at reversing the current situation are implemented, emissions from transport will continue to be a threat. International organizations have taken a lead in arranging meetings to search for possible solutions to the problem. One of the most remarkable conferences arranged to draw attention to global warming being worsened by the emissions of greenhouse gases (GHGs) was the Framework Convention on Climate Control in 1992. It was signed by 150 nations and all developed countries decided to bring down their CO₂ emissions by the year 2000 to the level in 1992. In addition to the conferences arranged to set targets in the reduction of GHGs, panels organized by various bodies have concentrated particularly on pollution created by the use of cars. For instance, a panel jointly organized by OECD and IEA in 1990 drew attention to the problem. Another example is Auto 2000 organized by the Commission of the European Community (CEC) in 1992. In both panels the possible solutions to increasing petrol consumption and air pollution by cars were discussed.

In recent years, various bodies concerned about the environment have suggested that existing taxes on goods that damage the environment should be used as a tool to preserve the environment. The aim of raising or using existing taxes on such goods is to induce consumers to switch their consumption towards more environmentally-friendly goods rather than providing new sources of revenue. Therefore, the increase of existing taxes on an environmental basis received a new definition: eco-taxes (OECD (1997)). According to the definition given by OECD, eco-taxes differ from those which mainly aim to bring more revenue to the budget. From this, it should be said that these taxes are not fiscal. The purpose of imposing such taxes is to make a change in people’s consumption pattern.

Having realized the negative impacts of transport on the environment and human health, measures aimed at reducing the use of private cars have been, and are being, taken in a number of countries. Policies designed to discourage people from making journeys by cars are attracting attention, particularly in large cities suffering from congestion and poor air quality for example Athens, Singapore and Hong Kong (see CEC (1990), Smith (1992), Phang and Toh (1997), McCarthy and Tay (1993), Hau (1990) and Morrison (1986)). Although the main factor giving rise to an increase
in fuel consumption is the increasing number of cars, there are other factors significantly affecting the amount of fuel consumed by cars such as the material used in the production of cars, engine type, traffic conditions and driver behaviour. Journeys taking less than 5 km made in urban areas constitute about 75 per cent of passenger journeys (OECD/IEA (1990)). Since traffic conditions in urban areas involve traffic jams and short trips, cars tend to consume more fuel there because of frequent braking and accelerating. According to a study by the German Motorists’ Organisation, the fuel consumption of a medium-sized family car at the beginning of a journey is about 40 litters/100 km. After 4 km it falls to its average level of about 8 litres/100 km (Schallabock (1992)). The speed at which a journey is made substantially affects fuel consumption. Fuel consumption is lowest at about 40-50 mph and 30-50 mph for petrol and diesel engined cars respectively. At low and very high speed fuel consumption will be higher (see for further reading Royal Commission (1994)). It should also be noted that emissions from cars are substantial at very low speeds as a result of fuel consumption. For instance, CO emissions are twenty times higher at 3 mph than at 62.5 mph. Likewise, HCs emissions are fifteen times higher (Crawford and Smith (1995)). From this, it can be seen that in congested areas cars tend to consume more fuel due to very low speed and this results in more pollution.

It is widely believed that one of the most effective ways of reducing fuel consumption is to minimize mileage driven. Increasing the variable cost of car use might be helpful in achieving a reduction and one way to do this is to raise the existing tax on fuel. For instance, when the oil crises arose in 1973 and 1980, petrol consumption by the transport sector as well as by other sectors went down as a result of increased prices in the UK (DOT (1984)). The distance travelled by cars, vans and taxis decreased in the year following the first oil crisis (DOT (1978)). In contrast, after the second oil crisis the distance travelled continued to rise (CSO (1989)). Interestingly, the oil crises did not have an impact on the engine size of cars purchased. The shift from cars with small engines to those with large engines continued after the crises (DOT (1978, 1985)). After the crises in other countries a drop in distance traveled was seen. The trend has shown an increase as a result of decreasing fuel prices in 1986 (Schipper (1995)).

Since people are unlikely to alter their habits voluntarily, it is necessary for the government to intervene in order to overcome the pollution being created by cars. By and large, government intervention is via tax instruments. In the UK motor vehicles are subject to three main taxes. VAT is levied on the initial purchase price, there is an annual Vehicle Excise Duty (VED) and fuel is also taxed. The VED on cars is related to engine size (cars having an engine size less than 1100cc are taxed at a lower rate) and to age (cars over 25 years old are tax exempt). For heavy goods vehicles VED is dependent on features such as the number of axles and weight. The annual tax in Germany was designed to encourage owners to make a switch towards more environmentally friendly cars, with a lower tax on those meeting required standards (Smith (1995), p. 96). Differentiation of the annual tax is also seen in Turkey but here it is old cars that are subject to lower taxes. This is based on distributional concerns rather than environmental ones.
If pollution is considered a problem, then measures taken to cope with it should aim to reduce the use of cars. Nevertheless, reducing the use of private vehicles might call restrictive policies that are difficult to implement. In such cases, a standard argument is that the demand for private vehicles should be shifted towards fuel-efficient cars through tax instruments. VED, an annual tax, is a fixed charge on car ownership whilst fuel taxation is a variable cost. Consequently, increasing VED principally affects the car ownership decision and increasing fuel taxes affect the use of cars and mileage driven. Differentiating VED according to the features of cars, such as engine size and weight, might lead consumers to make a change in their consumption pattern. As a report by OECD (1993) indicates, the role of tax differentiation is primarily its incentive impact and it is often used in a budget neutral manner. A standard argument, found for example in Pearson and Smith (1990), is that VED should be differentiated in favour of more environmentally-friendly cars. As the discussion of Section 2 makes clear, environmentally-friendly can be interpreted as smaller and more fuel-efficient. This can also be done through the VAT on purchase or possibly the fuel tax. The latter, though, requires considerably more administrative effort to implement. This point is discussed further below.

The intention of the paper is to assess the validity of the argument that taxes should be differentiated in favour of environmentally-friendly cars. To do this, it is necessary to construct a model which captures the reasons why some consumers choose small (meaning low fuel consumption and hence environmentally friendly) cars and others large (meaning high fuel consumption so environmentally unfriendly). The natural framework with which to work is one of vertical product differentiation. Such models assume that products can be ranked according to their quality level with all consumers agreeing with this ranking (see Ireland (1988)). This is also the case in the model of this paper but with one significant difference: the cost of a car is not just its purchase price but also its continuing running costs. It is assumed that high quality means large and less fuel efficient, hence high quality is offset by higher running costs. Although this adds another dimension to the vertical product differentiation model, the standard conclusion that high quality is purchased by consumers with high income still holds.

Given the fact that the high income consumers purchase less environmentally-friendly cars, it would seem that the environmental arguments for taxation would be reinforced by equity ones. That is, the tax on large cars could be used to subsidize small cars and thus redistribute from rich to poor as well as reducing pollution. The main point of the paper is to show that this need not be true and, in fact, it may be optimal to tax small cars despite their environmental advantage. The explanation for this surprising conclusion is founded directly in the use of vertical product differentiation as an explanation for the type of car purchased.

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1 Some advocate that the tax differentiation should also be made in favour of cars with diesel engines, since they consume less fuel. However, cars with diesel engines are not entirely pollution free. In terms of some pollutants such as particulate cars with petrol engines are superior to those with diesel engines. In addition a petrol car fitted with a three-way catalytic converter emits less NOx than a comparable diesel car. Our formal model considers only petrol cars but it could be extended to incorporate diesel.
Section 2 of the paper discusses the link between environmental friendliness, measured by fuel consumption, and car size. This is used to justify the use of a model in which environmentally-friendly cars are viewed as being small and low quality. The model is described in Section 3 and the equilibrium without taxation is constructed. Taxes are introduced in Section 4 and the effects of different forms of differentiation are considered. A numerical example is analyzed in Section 5. Section 6 discusses the extension to a continuous quality variable. The final section offers some conclusions.

2. Fuel consumption and quality

It is well known that cars with larger engines consume more fuel than those with small engines. This fact can be confirmed by a brief review of the motoring press, as can the observation that larger-engined and more-powerful cars are also more expensive. Despite this, recent years have witnessed a shift in demand towards larger and more powerful cars. This is especially the case in most European countries where the average new cars is now more powerful than was the case in the past. Over the years, particularly after the second oil crisis, there has also been an increase in the weight of cars in Europe (Schipper (1995)). Table 1 shows the number of cars registered in Great Britain in relation to engine size from 1987 to 1994. As can be seen from the table, there has been an increase in the number of cars with all engine sizes apart from those under 700 cc. The biggest increase has been witnessed in the cars with engines between 1,800 and 2,000 cc and the second biggest increase has been in cars with engines between 1,500 and 1,800 cc. Since these figures given the total stock of cars, the change in the flow is even more pronounced.

Table 2 provides evidence of the strong link between the quantity of fuel consumed by cars and their weight. This data comes from experiments conducted by the Peugot. As can be seen from the table, whilst model 305, weighting 940 kg, consumes 6.2 litre for 90 km/h, Vera +, weighting 700 kg, burns 3.5 litre for the same distance. Note that other features of cars concerning fuel consumption are not shown in the table, since the main aim is to show how the weight of cars affects fuel consumption.

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2 One objection to this claim is that old cars are generally more polluting than new cars. This is an important issue but it is not one that can be fully address in our static model.

3 The converse was the case in the US where legislation required the average fuel consumption of the cars sold by each manufacturer to fall. However, the growth in relative sales of four wheel drive utility vehicles has recently reversed the trend.
Table 1
Private and light goods vehicles: body type cars within private and light goods by engine size (thousands)

<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>700 cc</td>
<td>1,000</td>
<td>2,022</td>
<td>2,145</td>
<td>2,205</td>
<td>2,215</td>
<td>2,163</td>
<td>2,084</td>
<td>1,998</td>
<td>1,905</td>
</tr>
<tr>
<td>1,000 cc</td>
<td>1,200</td>
<td>2,154</td>
<td>2,201</td>
<td>2,215</td>
<td>2,226</td>
<td>2,198</td>
<td>2,207</td>
<td>2,227</td>
<td>2,261</td>
</tr>
<tr>
<td>1,200 cc</td>
<td>1,500</td>
<td>5,111</td>
<td>5,259</td>
<td>5,361</td>
<td>5,418</td>
<td>5,358</td>
<td>5,349</td>
<td>5,330</td>
<td>5,337</td>
</tr>
<tr>
<td>1,500 cc</td>
<td>1,800</td>
<td>4,850</td>
<td>5,279</td>
<td>5,641</td>
<td>5,872</td>
<td>5,944</td>
<td>6,025</td>
<td>6,129</td>
<td>6,276</td>
</tr>
<tr>
<td>1,800 cc</td>
<td>2,000</td>
<td>1,718</td>
<td>1,920</td>
<td>2,162</td>
<td>2,352</td>
<td>2,465</td>
<td>2,622</td>
<td>2,841</td>
<td>3,088</td>
</tr>
<tr>
<td>2,000 cc</td>
<td>2,500</td>
<td>710</td>
<td>736</td>
<td>747</td>
<td>744</td>
<td>726</td>
<td>719</td>
<td>722</td>
<td>759</td>
</tr>
<tr>
<td>2,500 cc</td>
<td>3,000</td>
<td>448</td>
<td>476</td>
<td>501</td>
<td>509</td>
<td>498</td>
<td>489</td>
<td>482</td>
<td>486</td>
</tr>
<tr>
<td>3,000 cc</td>
<td>294</td>
<td>307</td>
<td>314</td>
<td>315</td>
<td>306</td>
<td>305</td>
<td>312</td>
<td>313</td>
<td>1.5</td>
</tr>
</tbody>
</table>


Table 2
Fuel consumption by Peugeot 305, Vera and Vera+

<table>
<thead>
<tr>
<th></th>
<th>305</th>
<th>VERA</th>
<th>VERA+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight (kg)</td>
<td>940</td>
<td>752</td>
<td>700</td>
</tr>
<tr>
<td>Fuel Consumption (litre)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90 km/h</td>
<td>6.2</td>
<td>4.2</td>
<td>3.5</td>
</tr>
<tr>
<td>120 km/h</td>
<td>8.4</td>
<td>5.6</td>
<td>5</td>
</tr>
<tr>
<td>Urban</td>
<td>8.9</td>
<td>6.3</td>
<td>5.2</td>
</tr>
<tr>
<td>Average</td>
<td>7.8</td>
<td>5.4</td>
<td>4.6</td>
</tr>
</tbody>
</table>


As can be seen from Table 3, as well as weight, engine type affects fuel consumption by cars. Even if the weights of cars are the same, diesel-engined cars burn less fuel. Model Eve with a petrol engine, for instance, consumes 4.1 litre fuel for 90 km/h but the same model with a diesel engine burns 3.5 litre for the same distance. Diesel engines contribute to reducing CO₂, since they do not consume fuel as much as petrol ones which is why switching from petrol-engined cars to diesel-engined cars might be thought to be a successful policy for cutting CO₂ emissions from transport. On the other hand, diesel engines cannot achieve NOₓ emissions levels which are as low as those for petrol ones. Therefore, an increase in the number of diesel-engined cars in use will not be an effective tool in overcoming pollution problem.

As the above show, lighter cars with smaller engines consume less fuel. However, as a result of increasing disposable income, consumers have been demanding larger and more-comfortable cars which contribute more to air pollution. If consumers are to switch their demand towards more environmentally-friendly cars then government intervention is required. The aim now is to address the form that this intervention should take.

Table 3
### Fuel consumption by diesel- and petrol-engined cars

<table>
<thead>
<tr>
<th></th>
<th>R18</th>
<th>EVE</th>
<th>EVE+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight (kg)</td>
<td>920</td>
<td>845</td>
<td>845</td>
</tr>
<tr>
<td>Engine</td>
<td>Petrol</td>
<td>Petrol</td>
<td>Diesel direct injection turbocharged</td>
</tr>
<tr>
<td>1400 cm³</td>
<td>1100 cm³</td>
<td>1600 cm³</td>
<td></td>
</tr>
<tr>
<td>Max. speed</td>
<td>150 km/h</td>
<td>157 km/h</td>
<td></td>
</tr>
<tr>
<td>Fuel consumption</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90 km/h</td>
<td>6.3</td>
<td>4.1</td>
<td>3.5</td>
</tr>
<tr>
<td>120 km/h</td>
<td>8.4</td>
<td>5.5</td>
<td>4.4</td>
</tr>
<tr>
<td>Urban</td>
<td>9.4</td>
<td>6.6</td>
<td>5.8</td>
</tr>
<tr>
<td>Average</td>
<td>8.0</td>
<td>5.4</td>
<td>4.6</td>
</tr>
</tbody>
</table>


### Vertical differentiation

The previous section has provided evidence to illustrate the widely-accepted claim that smaller, lighter cars are environmentally more friendly than larger, heavier ones. To assess the policy of differential taxation for cars of different characteristics, it is necessary to have a model that explains why different consumers purchase different types of car. The most significant factor in this is that, running costs aside, large cars are regarded as fundamentally superior. There may be some exceptions to this, but generally the greater performance, carrying capacity, improved comfort and higher quality components are sought-after characteristics. Although there are some elements of horizontal differentiation involved in the analysis of cars, it is clear that vertical differentiation is relevant across one dimension of choice. The purpose of this section is to introduce a model of vertical differentiation and to analyze some of its implications.

The standard definition of vertical product differentiation is that if low and high quality versions of the same product are sold at the same price, all consumers prefer to buy high quality. This definition cannot be applied directly in the case of cars since the running costs must be taken into account as well as the purchase price. Even if two cars are sold at the same price, some consumers may prefer the low quality model if it is sufficiently cheaper to run. The definition of vertical differentiation must then be extended to: holding consumption of other goods constant, all consumers prefer the high quality to the low quality. Effectively, the quality comparison is made adjusting income to take account of running costs. As will be shown, it remains a feature of this extended form of vertical differentiation that low quality cars are purchased by low-income consumers and high quality by high income.

The economy that is considered in the following two sections has a continuum of consumers and two quality levels. In addition to the quality-differentiated car, there is also available another “composite” commodity. All consumers have the same preferences and incomes are exogenous. Each consumer buys at most one car. The use of a car creates a negative externality through its pollution etc. The high quality car produces more externality per unit of distance travelled through factors such as the higher fuel consumption discussed above. This externality is not internalized by consumers, so the unregulated equilibrium of the economy is inefficient.

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4 The extension to a quality continuum is considered in Section 6.
Let $q(s)$ represent the price of a car of type $s$ and $r(s)$ be the running cost per unit distance. Let $s = 1$ and $s = 2$ denote the quality level of the small car and the large car respectively. It is assumed that the prices and running costs satisfy

$$q(2) > q(1),$$

$$r(2) > r(1),$$

so that the high-quality car is more expensive to buy and run than the low-quality car. Higher running costs can be due to both the higher fuel consumption and more expensive service charges, but are interpreted here as arising solely from the cost of fuel per unit of distance.

Now consider a consumer whose preferences can be represented by the utility function

$$U = U(x, y; s, e),$$

where $x$ is a composite of “other goods”, $y$ is distance travelled and $e$ represents the externality arising from the total use of cars in the economy. By necessity, $y$ can only be positive if a car is purchased. This utility function satisfies Assumption 1.

Assumption 1

(i) $U(x, y; s, e)$ is concave in $x$ and $y$;

(ii) $U(x, y; s, e) > 0 \forall x > 0, y > 0, \forall s, e$;

(iii) $U(x, 0; s, e) = 0, U(0, y; s, e) = 0$;

(iv) $U(x, y; 1, e) < U(x, y; 2, e) \forall x > 0, y > 0, e$.

The first, second and third of these conditions are standard. The fourth is the extended definition of vertical product differentiation: for equal levels of consumption, the higher quality car will be preferred.

The budget constraint of a consumer with income $M$ is

$$M = q(s) + r(s) y + x.$$  

(4)

To avoid the need to consider corner solutions, it is assumed that $M > q(1)$. Given prices and income, $s$, $y$ and $x$ are chosen to maximize (3) subject to (4) and taking $e$ as given. The constancy of $e$ from the perspective of an individual consumer is justified because each consumer is of measure zero in the continuum. This problem is solved in two stages. First, the maximal level of utility for each value of $s$ is found. This gives a derived utility $U = V(M - q(s), r(s); s, e)$, $s = 1, 2$. Second, these derived utilities are contrasted and that which is greater determines the chosen quality level.

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5 This can be converted to a price per gallon by dividing by miles per gallon.
The derivatives of the derived utilities are needed below. These are summarized in Lemma 1; the proof is a standard application of the envelope theorem and is omitted.

**Lemma 1.** For fixed $s$, the derivatives of the derived utility satisfy

$$
\frac{\partial V}{\partial M} = -\frac{\partial V}{\partial q(s)} = \alpha(M; s, e), \quad \frac{\partial V}{\partial r(s)} = -\alpha(M; s, e)y(M; s, e).
$$

where $\alpha(M; s, e)$ and $y(M; s, e)$ are the marginal utility of income and the demand for travel respectively conditional on income, quality and externality.

The consequence of contrasting the derived utilities is summarized in Lemma 2. This shows that it is high-income consumers that purchase high quality, a result consistent with the standard model of vertical differentiation without running costs (see Gabszewicz and Thisse (1979) and Myles (1988)).

**Lemma 2.** For fixed prices and level of externality, there is an income level, $M^* > 0$, such that:

(i) $V(M - q(1), r(1), 1, e) > V(M - q(2), r(2), 2, e)$ for $M < M^*$;

(ii) $V(M - q(1), r(1), 1, e) = V(M - q(2), r(2), 2, e)$ for $M = M^*$;

(iii) $V(M - q(1), r(1), 1, e) < V(M - q(2), r(2), 2, e)$ for $M > M^*$.

**Proof.** Consider an income level $q(1) < \tilde{M} < q(2)$. Then $y(2) = 0$ and, by (ii) and (iii) of Assumption 1, $V(\tilde{M} - q(1), r(1), 1, e) > V(\tilde{M} - q(2), r(2), 2, e)$. Next note that (iv) of Assumption 1 holds for all $x, y$. In particular, it holds as $M \to \infty$ and $x \to \infty, y \to \infty$ so

$$
\lim_{M \to \infty} V(M - q(1), r(1), 1, e) \geq \lim_{M \to \infty} V(M - q(2), r(2), 2, e).
$$

From (i) of Assumption 1, each derived utility must be a concave function of income. Since $V(M - q(1), r(1), 1, e)$ has been shown to be larger for a low income level and $V(M - q(2), r(2), 2)$ larger for high incomes, the concavity implies that they must cross at a unique point. This is the critical income level $M^*$. ||

The derived utility functions and the critical income level are illustrated in Figure 1. The interpretation is that if the income level of the consumer is below $M^*$ they will buy quality level 1 while they will buy quality level 2 if their income is above $M^*$. More importantly it follows from Lemma 2 that (generically) the marginal utility of income is a discontinuous function of income. It falls until income $M^*$ is reached, then increases discontinuously at $M^*$ and then falls again above $M^*$. Consequently,

$$
\lim_{M \uparrow \to M^*} \alpha(M; 1, e) < \lim_{M \downarrow \to M^*} \alpha(M; 2, e).
$$

---

6 It is possible for the two derived utilities to have identical gradients at the point of intersection. But this would not be robust to any change in $q$ and $r$, so would hold only on a set of measure zero in the space of prices and running costs.
This is shown in Figure 2.

Figure 1: Critical income level

Figure 2: The marginal utility of income

The unique value of $M^*$ satisfies the equation

$$V(M^* - q(1), r(1), s, e) = V(M^* - q(2), r(2), s, e). \quad (6)$$

The inequality in (5) ensures that the Implicit Function Theorem can be applied to (6) to give

$$M^* = M^*(q(1), q(2), r(1), r(2), e). \quad (7)$$

Lemma 3 captures the dependence of $M^*$ on its arguments.

**Lemma 3.** The function $M^*(q_1, q_2, r_1, r_2; e)$ satisfies:

$$\frac{dM^*}{dq(1)} < 0, \quad \frac{dM^*}{dq(2)} > 0;$$

and
\[
\frac{dM^*}{dr(1)} < 0, \frac{dM^*}{dr(2)} > 0.
\]

**Proof.** Using Lemma 1 and inequality (5). \\

Now consider a population of consumer. The income distribution is described by a density function \( f(M) \) which has strictly positive support on the interval \([\underline{M}, \bar{M}]\). The density satisfies Assumption 2.

**Assumption 2.**

(i) \( f(M) > 0 \) \( \forall M \in [\underline{M}, \bar{M}] \);

(ii) \( \int_{\underline{M}}^{\bar{M}} Mf(M) dM \leq \bar{M} < \infty \).

The division of the population according to their choice of quality level reflects the individual choices shown in Figure 1: consumers with income less than \( M^* \) buy the low quality, those with incomes above \( M^* \) buy high quality. Thus given a level of the externality, it is possible to find aggregate demand by integrating over the relevant sub-population.

The construction so far has been for a given value of the externality. To complete the model it is necessary to endogenize this. In the context of externalities from car use, the natural choice is to assume that the externality takes the form of Meade’s “atmosphere externality” (see Myles (1995) p. 337), so the total externality is dependent on the sum of individual contributions. This is a reasonable first approximation to production of atmospheric pollutants. Let the marginal contribution of the use of a car of quality \( s \) for 1 unit of distance be \( e_s \), where \( e_2 > e_1 \). The total level of externality is then

\[
e = e_1 \int_{\underline{M}}^{M^*} y_1 f(M) dM + e_2 \int_{M^*}^{\bar{M}} y_2 f(M) dM,
\]

where \( y_s = y_s(M - q(s), r(s), e) \), is the mileage driven in a car of quality \( s \) as a function of income, prices, and externality.

To prove that an equilibrium exists it is necessary to study the structure of (8). Since \( M^* \), \( y_1 \) and \( y_2 \) are dependent on \( e \), the right-hand side of (8) can be treated as a function, \( f(e) \), of \( e \). An equilibrium level of the externality, \( \hat{e} \), is a fixed point of the function and hence is defined by \( \hat{e} = f(\hat{e}) \). There is an equilibrium if \( f(e) \) has at least one fixed point. The proof that it does is given in Theorem 1.

**Theorem 1.** Under Assumptions 1 and 2 there is an equilibrium of the economy.
Proof. Let \( e = 0 \). Since either \( y_1(M_1 - q_1, r_1, 0) > 0 \) or \( y_2(M_2 - q_2, r_2, 0) > 0 \) (or both) it follows that
\[
f(0) > 0. \tag{9}
\]

Now define \( \tilde{M} \) by
\[
\tilde{M} \equiv \arg \max \left\{ e_1 \int \frac{M - q_1}{r_1} f(M) dM + e_2 \int \frac{M - q_2}{r_2} f(M) dM \right\},
\tag{10}
\]
and the resulting level of the externality by \( \tilde{e} \). By construction, \( \tilde{e} \) is the maximum level of externality that can be achieved when all income in excess of car purchase costs is spent on fuel. Then for all \( e \), \( f(e) < \tilde{e} \) since \( x > 0 \). Since \( \int \frac{M f(M) dM}{M} \) is finite, so must be \( \tilde{e} \). Hence,
\[
f(e) < e \quad \forall e \geq \tilde{e}. \tag{11}
\]
Combining (9) and (11), the continuity of the integrals then proves that there is at least one fixed point. \( \Box \)

Given that use is made below of comparative statics analysis, it is worth briefly addressing the uniqueness of equilibrium. In the general case, a sufficient condition for uniqueness is that \( f(e) \) is a contraction mapping (\(|f'(e)| < 1\)). But the structure here is more specific in that \( f(e) \) must satisfy (10) and (11). This weakens the sufficient condition to \( f'(e) < 1 \). To investigate when this might hold, note that from (8)
\[
f'(e) = \left[ e_1 y_1 - e_2 y_2 \right] f(M^*) \frac{\partial M^*}{\partial e} + e_1 \int \frac{\partial y_1}{\partial e} f(M) dM + e_2 \int \frac{\partial y_2}{\partial e} f(M) dM. \tag{12}
\]

From (12) it can be seen that uniqueness is guaranteed if the utility function is additive separable so \( U = u_1(x, y; s) + u_2(e) \). With this restriction, \( \frac{\partial M^*}{\partial e} = 0 \) and \( \frac{\partial y_i}{\partial e} = 0, s = 1, 2 \). Hence \( f'(e) = 0 \) and the uniqueness condition is satisfied.\footnote{This is the formulation used in the numerical work of Section 5.}

Alternatively, if \( \frac{\partial y_i}{\partial e} < 0 \) then it is sufficient for uniqueness that
\[
\left[ e_1 y_1 - e_2 y_2 \right] \frac{\partial M^*}{\partial e} < 0.
\]
The expression \( [e_1y_1 - e_2y_2] \) arises again below so it is worth a closer look. If \( e_1y_1 - e_2y_2 > 0 \) it is said that the *mileage reduction condition* holds. To understand this, it should be recalled that \( y_1 \) and \( y_2 \) are both evaluated at \( M^* \) and are, respectively, the mileages driven by the richest consumer to buy a low-quality car and by the poorest to buy high quality. It is always the case that \( y_1 > y_2 \). The mileage reduction condition means that \( y_2 \) is sufficiently below \( y_1 \) that the extra externality generated by the high-quality car is offset by the lower mileage of the *marginal* purchaser. This condition will prove to be important in the analysis below.

As this discussion has shown, there are numerous sets of conditions that will guarantee uniqueness. This provides support for the using comparative statics analysis in Section 4 without too many concerns because when the equilibrium is unique, it must also be continuous in its parameters. This completes the discussion of the basic model.

### 4. Taxation

The model set out in Section 3 can now be employed to discuss the design of the tax structure. In particular, if the VED can be differentiated between different qualities of car, is it always the case that the high quality car should be taxed more heavily? And would the same arguments imply that the fuel tax should be differentiated in the same way? The analysis of marginal tax reform from an initial position with no taxation will be employed to show that there are two factors at work in determining the answers to these questions. These are termed the *marginal utility effect* and the *externality effect*. As the analysis proceeds it becomes clear that these need not work in the direction expected.

#### 4.1 VED

Consider first the VED. Denote the tax levied on the car of quality \( s \) by \( T_s \). The VED raises the price of the car of quality \( s \) to \( q(s) + T_s \) so derived utility becomes \( V(M - q(s) - T_s, r(s), s, e) \). Social welfare is determined by the utilitarian function

\[
W = \int_{M^*}^{M} V(M - q(1) - T_1, r(1), e) f(M) dM + \int_{M^*}^{M} V(M - q(2) - T_2, r(2), e) f(M) dM.
\]

(13)

The major focus is placed upon which of the two qualities of car should be taxed at the highest rate. To make the comparison as clear-cut as possible, it is assumed that the taxes raise no net revenue. This is in agreement with the policy proposals discussed in Section 1. Since each consumer buys exactly one car, the taxes must satisfy the budget constraint

\[
\int_{M}^{M^*} T_1 f(M) dM + \int_{M^*}^{M} T_2 f(M) dM = 0.
\]

(14)
To find the effects of the taxes it is necessary to compute their effect upon the equilibrium of the economy. This is done by working from equation (8). At equilibrium, it follows that

\[ e = f(e, T_1, T_2), \]  

(15)

where

\[ f(e, T_1, T_2) = e_1 \int_{M}^{M^*} y_1(M - T_1)f(M) dM + e_2 \int_{M}^{M^*} y_2(M - T_2)f(M) dM . \]  

(16)

Provided \( f_e < 1 \) (which is the uniqueness condition), the implicit equation (15) can be solved to write

\[ e = e(T_1, T_2). \]  

(17)

Begin now at an initial position where \( T_1 = T_2 = 0 \) and consider the introduction of differential taxes \( dT_s, s = 1, 2. \) Using (14) and (17) it follows that \( dT_2 < 0 \) (and \( dT_1 > 0 \) by budget balance) if

\[ \int_{M}^{M^*} \alpha(M; 1)f(M) dM - W_e e_1 - \int_{M}^{M^*} \alpha(M; 2)f(M) dM - W_e e_2 < 0, \]  

(18)

where

\[ W_e \equiv \int_{M}^{M^*} V_e f(M) dM + \int_{M}^{M^*} V_e f(M) dM < 0. \]  

(19)

To interpret (18) it is best to begin with the case of no externality and so obtain an insight into one of the factors at work: the marginal utility effect. With no externality \( W_e \equiv 0 \) and (18) reduces to

\[ \bar{\alpha}(1) - \bar{\alpha}(2) < 0, \]  

(20)

where \( \bar{\alpha}(s) \) is the mean marginal utility of income of those consuming quality \( s. \) With concave utility and no quality differential, it has to be the case that marginal utility of income falls with income so that the analogue of (20) could never hold. With the quality differential, it becomes perfectly possible for (20) to hold because of the kink in the derived utility function encapsulated in condition (5) and Figures 1 and 2. In other words, the existence of the high quality alternative raises the marginal utility of its marginal purchases above that of the marginal purchasers of low quality. If this effect is sufficiently marked it can make the mean marginal utility of the
purchasers of high quality (who are the relatively rich) greater than that of purchasers of low quality (the relatively poor). When this happens, the tax system will want to subsidize the high quality product.

The outcome of this discussion is that the marginal utility effect does not necessarily imply the lower taxation of the low-quality car. If fact, it is perfectly possible for it to imply the converse. The simple intuition that high-quality cars must be taxed more highly because they are consumed by the higher income consumers is not an accurate guide to policy.

Now that the marginal utility effect is understood, it is possible to reintroduce the externality. Since \( W_e < 0 \), a sufficient condition (though not necessary) for the externality to increase the likelihood of subsidizing high quality is \( e_{gT_1} > 0 > e_{gT_2} \). Calculating these terms,

\[
e_{gT_1} := -\left[ e_1 y_1 - e_2 y_2 \right] f(M^*) \frac{\alpha(1)}{\alpha(2) - \alpha(1)} - e_1 \int_0^{M^*} \frac{\partial y_1}{\partial M} f dM ,
\]

and

\[
e_{gT_2} := \left[ e_1 y_1 - e_2 y_2 \right] f(M^*) \frac{\alpha(2)}{\alpha(2) - \alpha(1)} - e_2 \int_0^{M^*} \frac{\partial y_2}{\partial M} f dM .
\]

If the mileage reduction condition holds (so \( e_1 y_1 - e_2 y_2 > 0 \)), then \( e_{gT_1} \) is negative and \( e_{gT_2} \) is unsigned. The externality then raises the likelihood of subsidizing high quality if travel demand of those consuming high quality is income inelastic (thus making \( e_{gT_2} \) positive). Expressed simply, since the high-quality car is more expensive it leaves its purchasers with a lower disposable income. Consequently, its marginal purchasers will choose to cover less mileage than the marginal purchasers of the low-quality car. Despite the greater externality per mile, the total externality produced by the high-quality car may be less than that of the low quality. Hence the externality effect of taxation can also operate in a direction converse to simple intuition.

The most important conclusion from the analysis is that there are situations under which \( dT_2 < 0 < dT_1 \). If the optimization problem is concave, the derivation extends to concluding that the optimal tax rates satisfy \( T_2 < 0 < T_1 \). In any case, the finding that the low-quality car may be taxed more runs counter to the suggestion of simple intuition that both distributional effects and externality effects work in the direction of high-quality cars being taxed more. As shown, this intuition can be wrong on both counts. Although the outcome is not entirely clear cut at this level of generality, the formal modelling has raised issues that would not be apparent without it.

4.2 Fuel tax
When a fuel tax is considered, it is interesting to see if the same kinds of arguments apply. To this end a differentiated fuel tax is now analyzed which can be levied at a quality-dependent rate. Clearly there are practical issues involved in implementing such a tax but none which are insurmountable. So, although it would not be as natural to implement as a differentiated VED, it can be envisaged as functioning in practice. In the two-quality setting used here, the problems of implementation are much reduced.

Denoting the \( (ad\ valorem)\) tax on fuel for a car of quality \( s \) by \( \tau_s \), the derived utility becomes \( V(M - q(s)[1 + \tau_s]r(s)s,e) \) with

\[
\frac{\partial V}{\partial \tau_s} = -\alpha(M; s,e)r(s)y_s. \tag{23}
\]

The critical income level (7), the level of social welfare (13) and the determination of the total externality (8) now become dependent on the tax-inclusive price \( [1 + \tau_s]r(s) \). The level of government revenue is

\[
R = \int_{\tau_s}^{M^{*}(\tau_s)} \tau_s r(1) y_1 f(M) dM + \int_{\tau_s}^{M^{*}(\tau_s)} \tau_s r(2) y_2 f(M) dM . \tag{24}
\]

As before, the exercise is repeated beginning in an initial position with no taxation and determining the direction in which taxes should move in order to raise welfare. This provides the result that \( d\tau_1 > 0, d\tau_2 < 0 \) if

\[
\frac{\int_{\tau_s}^{M^{*}} \alpha r(1) y_1 f dM - W e_{\tau_1}}{\int_{\tau_s}^{M^{*}} r(1) y_1 f dM} - \frac{\int_{\tau_s}^{M^{*}} \alpha r(2) y_2 f dM - W e_{\tau_2}}{\int_{\tau_s}^{M^{*}} r(2) y_2 f dM} < 0 . \tag{25}
\]

An interpretation of this condition can again be developed by beginning with the case of no externality. Then (25) simply becomes

\[
\frac{\overline{\alpha(1)} y_1}{\overline{y}_1} - \frac{\overline{\alpha(2)} y_2}{\overline{y}_2} < 0 , \tag{26}
\]

or that the mean marginal utility of income generated from mileage is higher for the high quality than for the low. In a two class economy for which \( \alpha(1) \) is constant for one class and \( \alpha(2) \) constant for the other, (26) reduces

\[
\alpha(1) - \alpha(2) < 0 , \tag{27}
\]

which is of the same form as (20) but with the marginal utilities evaluated at the income levels of the two classes not at the critical income \( M^{*} \).
To see the effect of the including the externality, return to (25). The effects of the fuel tax on the externality are given by

\[ e_\tau := -[e_1 y_1 - e_2 y_2] f(M^\ast) \frac{\alpha(1)}{\alpha(2) - \alpha(1)} r(1) y_1 + e_1 \int_{M^\ast} \frac{\partial y_1}{\partial \tau^1} f dM, \]  

(28)

and

\[ e_{\tau^2} := [e_1 y_1 - e_2 y_2] f(M^\ast) \frac{\alpha(2)}{\alpha(2) - \alpha(1)} r(2) y_2 + e_2 \int_{M^\ast} \frac{\partial y_2}{\partial \tau^2} f dM. \]  

(29)

The fact that \( W_\tau < 0 \) means that a sufficient condition for the externality effect to work in the direction of a lower tax on high quality if \( e_{\tau^1} > 0 > e_{\tau^2} \). From (28) and (29) it can be seen that this is most likely if the mileage reduction condition holds and demand for travel is price inelastic. In detail, if the mileage reduction condition holds then \( e_{\tau^1} < 0 \) but \( e_{\tau^2} \) is not signed. Consequently, the externality effect may again operate in the direction of subsidization for high quality if the mileage reduction condition holds.

This analysis of fuel taxes has very much echoed that of VED. The major distinction is that fuel taxes are distortionary at all points on the income range whereas the VED only distorts at the margin between high and low quality. Otherwise the conclusions are the same: it is possible for both the marginal utility effect and the externality effect to work in the direction of a lower fuel tax on high quality cars relative to that on low quality.

**4.3 The first-best**

Since the VED operates as a transfer of income and the fuel tax as a Pigouvian tax, it might be thought that the combination of these instruments could achieve the first-best outcome. It is easily shown that this is not the case. Firstly, a differentiated VED is not a lump-sum tax since it is conditional on quality of car owned which is choice variable for the consumers, not an unalterable characteristic. Secondly, the income transfers necessary for the first-best need to more sophisticated than can be achieved through the VED.

To see this latter point, return to Figure 2. A first-best allocation of income would involve finding a common tangent to the two derived utility functions (giving two points where marginal utility of income was equal) and allocating the consumers to these points in a way that balanced the budget. Hence, an income transfer system that achieved the first-best would have to take the initial distribution of income and focus it on two points. All that the VED can do is to transfer income from consumers of low quality to consumers of high quality (and vice versa) which is not sufficient.

**4.4 Summary**
The levying of a higher rate of VED on higher quality cars has become a standard policy suggestion. Since high quality cars are consumed by the rich and produce greater externalities, it would appear that both distributional and externality factors support these proposals. When the choice of car quality is modelled as one involving vertical differentiation, both of these arguments are undermined and may in some cases be entirely reversed. The same arguments also apply to a differentiated fuel tax. Overall, the policy proposals are not built on the solid ground they first appear to be on.

5. Ranking of policies

The previous section has isolated the factors that lie behind the determination of relative tax rates. In particular, it has identified the role of the marginal utility and externality effects and demonstrated that both these factors can lead in the direction of the higher quality car being subsidized. However, this analysis is limited in that it cannot make contrasts between policies—such as determining whether the optimal VED dominates the optimal fuel tax. To do this a specific example is now analyzed. This will give some insight into the magnitudes involved, the range of parameter values for which converse results can arise and the welfare ranking of alternative policies.

The utility function is specialized to

\[ U = \log(sy) + \log x - e, \quad (30) \]

where the additive separability of the externality effect in (30) means that the demands for \( x \) and \( y \) are independent of \( e \). Referring back to (8), this implies that the right-hand side is independent of \( e \), so that the fixed point argument does not need to be employed to solve for equilibrium. The derived utility corresponding to (30) is given by

\[ V(s) - e = \log \left( \frac{M - q(s) - T_s}{2s} \right) + \log \left( \frac{M - q(s) - T_s}{2} \right) - e. \quad (31) \]

Hence the critical income level, \( M^* \), satisfies

\[ \log \left( \frac{M^* - q(1) - T_1}{2r(1 + \tau_1)} \right) \left( \frac{M^* - q(2) - T_2}{2r(2 + \tau_2)} \right) = \log \left( \frac{M^* - q(1) - T_1}{2} \right) \left( \frac{M^* - q(2) - T_2}{2} \right). \quad (32) \]

For simplicity, it is assumed that there are two consumers. One is rich and buys the high-quality car while the other one is poor and buys the low quality. The tax rates considered all ensure that these choices remain optimal. Finally, it is assumed that the externality is generated according to the relationship

\[ e = \frac{1}{2} [a(1) \log y(1) + a(2) \log y(2)]. \quad (33) \]
This is not the Meade externality defined in (8) but is chosen to ease the calculations. Putting these assumptions together, the level of social welfare is

\[ W = \sum_{s=1}^{2} \left[ \log(sy(s)) + \log(x(s)) - a(s)\log(y(s)) \right]. \] (34)

It is now possible to examine the tax rates implied by the maximization of social welfare function and the maximized level of welfare. Four different policy regimes are considered. The first considers the efficient choice of quantities whilst respecting the initial budget constraints. Differentiated VED is considered next and finally the fuel tax, both uniform across qualities and differentiated.

**Case 1. The “efficient” solution**

The “efficient” solution involves the quantities being chosen to maximize social welfare whilst respecting the individual budget constraints. The externality is internalized but there is no income redistribution. From (34), the efficient quantities, for \( s = 1, 2 \) satisfy

\[ y(s) = \frac{[1 - a(s)][M(s) - q(s)]}{2 - a(s)f(s)}, \] (35)

and

\[ x(s) = \frac{M(s) - q(s)}{2 - a(s)}. \] (36)

**Case 2. Differentiated VED**

The analysis of taxation assumes that all revenue is redistributed to the consumers by the government in the form of a lump-sum subsidy. With the differentiated VED, the implied budget constraint for the government is

\[ T_1 + T_2 - 2\xi = 0, \] (37)

where \( \xi \) is the lump-sum subsidy which returns the revenue from the VED to the consumers.

Using the budget constraint (37), the level of social welfare is

\[ W = \sum_{s=1,2} \left[ \log\left( \frac{s}{2r(s)} \left[ M(s) - q(s) - \frac{T_{s'}}{T_{s'} + T_{s'}} \right] \right) + \log\left( \frac{1}{2} \left[ M(s) - q(s) - \frac{T_{s'}}{T_{s'} + T_{s'}} \right] \right) - a(s)\log\left( \frac{1}{2r(s)} \left[ M(s) - q(s) - \frac{T_{s'}}{T_{s'} + T_{s'}} \right] \right) \right]. \] (38)

It can be seen directly from (38) that a non-differentiated VED \((T_1 = T_2)\) will have no effect upon social welfare. This follows from the assumption of the lump-sum return of taxation: since both of the consumers buy a car, the VED is simply a
lump-sum tax. It then just cancels with the revenue redistributed by the government. Only if the tax is redistributed unevenly or through a distortionary instrument, can it have an effect. But neither of these features is central to the analysis of the effects of the taxes upon car use, so they are not pursued further.

Returning to the differentiated VED, the optimal values of \( T_1 \) and \( T_2 \) solve

\[
T_1 = \frac{[a(1)2 - 4M(2) - q(2)] + [4 - a(2)2M(1) - q(1)] + [4 - a(1) - a(2)]T_2}{4 - [a(1) + a(2)]},
\]

(39)

\[
T_2 = \frac{[4 - a(1)2M(2) - q(2)] + [a(2)2 - 4M(1) - q(1)] + [4 - a(1) - a(2)]T_1}{4 - [a(1) + a(2)]}.
\]

(40)

Inspection of (39) and (40) shows that \( T_1 \) and \( T_2 \) are undefined if \( a(1) + a(2) = 4 \). So, it is assumed from this point that \( a(1) + a(2) < 4 \). This simply restricts the externality effect not to be too excessive. The optimal taxes are also colinear, so that a unique solution cannot be found. In fact there is a continuum of solutions, and any one of these can be selected by arbitrarily fixing a value for the one of the tax rates. This is what will be done in the simulations reported below. The reason for this outcome is that both the VED and the method of returning revenue are non-distortionary so it is only the difference between the two rates of VED that is relevant.

For now the focus is placed on the conditions that imply the tax on the high quality car is less than that on the low quality. Using the two solutions

\[
T_2 - T_1 = \frac{[4 - a(1)2M(2) - q(2)] + [a(2)2 - 4M(1) - q(1)]}{4 - [a(1) + a(2)]}.
\]

(41)

Solving the condition, the larger car is subsidized if

\[
M(2) < \frac{2 - a(2)}{2 - a(1)} M(1) + \left[ q(2) - \frac{2 - a(2)}{2 - a(1)} q(1) \right].
\]

(42)

Clearly, subsidization of the larger car is most likely when: \( M(1) \) and \( M(2) \) are close to \( M^* \); \( q(2) \) is large relative to \( q(1) \); and \( a(1) \) and \( a(2) \) are of similar, small magnitude. The first two of these factors work in the direction of raising the marginal utility of consumer 2 relative to that of consumer 1. The effect of the externality is to reduce the negative environmental impact of the larger car relative to the smaller car.

Table 4 shows the amount of VED levied on the low-quality car when the tax on the high quality is normalized. In order to see the link between \( M(1) \) and \( T_1 \), the value of \( M(1) \) is increased by keeping the values of all the other variables constant in the first three rows. As \( M(1) \) is increased \( T_1 \) increases relative to \( T_2 \) and becomes greater than \( T_2 \) in row 3. This latter case confirms the possibility of the high quality
having a lower tax identified in Section 4. The same exercise is then repeated by
assuming that the tax on the large car is normalized at 1. This gives rise to an equal
increase in $T_i$, a fact that is due to the colinearity between the two. The next step is to
raise the values of both $M(1)$ and $q(1)$ without changing the difference between the
two. This does not have an impact on $T_i$. Finally, the values of both $M(1)$ and $q(1)$
are increased and the difference between varied. In this as the difference
between $M(1)$ and $q(1)$ increases, $T_i$ increases. Identical properties would hold if $T_i$
were normalized and the consequent value of $T_2$ calculated.

Table 4

<table>
<thead>
<tr>
<th>Differentiated VED: $T_2$ normalized</th>
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<tbody>
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<td>$M(1)$</td>
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<td>9</td>
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<tr>
<td>8</td>
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<tr>
<td>9</td>
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</table>

Case 3. Uniform fuel tax
After the imposition of a fuel tax, the budget constraint for the government is

$$\tau r(1)y(1) + \tau r(2)y(2) = 2\xi.$$  \hspace{1cm} (43)

Working through the individual choice problems and then maximizing social welfare
by the choice of $\tau$ yields

$$[1 - a(1)] \left[\frac{5[M(1) - q(1)] + [M(2) - q(2)]}{(4 + 5\tau)[M(1) - q(1)] + \tau[M(2) - q(2)]}\right] - [2 - a(1) - a(2)] \left(\frac{1}{2 + \tau}\right) =$$

$$- [1 - a(2)] \left[\frac{5[M(2) - q(2)] + [M(1) - q(1)]}{(4 + 5\tau)[M(2) - q(2)] + \tau[M(1) - q(1)]}\right] + [2 - a(1) - a(2)] \left(\frac{1}{1 + \tau}\right).$$  \hspace{1cm} (44)

Table 5 shows the optimal values of the fuel tax. As in the case of VED, in
order to trace the effect of income levels on the fuel tax, $M(2)$ is increased without
making a change in the other variables shown in the table in the first three rows.
Unsurprisingly, as $M(2)$ increases, $\tau$ increases. Then $q(2)$ is increased but keeping
the difference between $M(2)$ and $q(2)$ fixed. This has no impact on $\tau$. As the
difference increases, $\tau$ increases. The same exercise is repeated for $M(1)$ and $q(1)
to see the effect of \( M(1) \) on \( \tau \). Unlike the difference between \( M(2) \) and \( q(2) \) as the difference between \( M(1) \) and \( q(1) \) increases, \( \tau \) decreases. It is also worth noting that when \( M(1) - q(1) = M(2) - q(2) \), \( \tau \) is zero.

**Table 5**  
Uniform fuel tax

<table>
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<tr>
<th>( M(1) )</th>
<th>( q(1) )</th>
<th>( a(1) )</th>
<th>( M(2) )</th>
<th>( q(2) )</th>
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<td>0.5</td>
<td>0.293</td>
</tr>
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</table>

**Case 4. Differentiated fuel tax**

A differentiated fuel tax leads to the budget constraint

\[
\tau_1 r(1)y_1 + \tau_2 r(2)y_2 = 2\xi. 
\]  

so

\[
\xi = \frac{\tau_1 [M(1) - q(1)I + \tau_2] + \tau_2 [M(2) - q(2)I + \tau_1]}{4I + \tau_1[I + \tau_2] - \tau_1[I + \tau_2] - \tau_2[I + \tau_1]}. 
\]  

Substituting (46) into the definition of welfare and optimizing over \( \tau_1 \) and \( \tau_2 \) generates the results in Table 6. These show that as \( M(2) \) is increased holding \( M(1) \) constant, then \( \tau_2 \) rises relative to \( \tau_1 \). In all cases \( \tau_2 \) is greater than \( \tau_1 \), reflecting the fact that the conditions are more stringent for the converse case to arise with the differentiated fuel tax.

**Table 6**  
Differentiated fuel tax

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<thead>
<tr>
<th>( M(1) )</th>
<th>( q(1) )</th>
<th>( a(1) )</th>
<th>( M(2) )</th>
<th>( q(2) )</th>
<th>( a(2) )</th>
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Having determined these optimal taxes it is now possible to examine which policy leads to the highest level of welfare. The results of this analysis are reported in Table 7.

### Table 7
Comparison of policies

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<th>M(2)</th>
<th>q(2)</th>
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<td>3.470</td>
<td>3.654</td>
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</tbody>
</table>

SWM: market outcome; SWE: efficient outcome; SWT1: VED on low quality only; SWT2: VED on high quality only; SWU: uniform fuel tax; SWD: differentiated fuel tax

Several observations can be made about the results reported in Table 7. Firstly, it is interesting to see that SWU > SWE, so that the uniform fuel tax leads to welfare level greater that that achieved by the efficient allocation of commodities. This is because the fuel tax both lessens the externality by discouraging the use of cars and leads to some income redistribution. Secondly, SWT > SWU so the differentiated VED is preferable to a uniform fuel tax. Finally, which of the differentiated taxes generates the highest welfare level depends on the degree of income inequality. When this is large, the differentiated VED is best.

### 6. A quality continuum

The model so far has been concerned with an economy in which there are only two quality levels. The discrete that occurs between qualities when the critical income is passed leads to a discontinuity in the marginal utility of income. When applied to tax policy, this discontinuity can have the consequence of making it optimal to subsidize the higher quality car despite it being purchased by higher income consumers and causing relatively more environmental damage. With quality as a continuum, this
discontinuity cannot occur which raises the question of whether the argument for subsidizing the higher quality is still valid.

To investigate this issue, let there be available a compact interval of quality levels $[s, \bar{s}]$. The price and running cost for each quality level are given by the continuous and strictly increasing functions $q(s), r : [s, \bar{s}] \rightarrow \mathbb{R}_{++}$ and $r(s), r : [s, \bar{s}] \rightarrow \mathbb{R}_{++}$. A consumer with income $M$ solves the optimization

$$\max_{\{x, y, s\}} U(x, y, s; e) \text{ subject to } M = q(s) + r(s)y + x,$$

which has necessary conditions

$$U_x - U_x[q' + r'y] = 0,$$

and

$$U_y - U_y r = 0.$$

The solutions $x(M)$, $y(M)$ and $s(M)$ generate a maximum value function

$$V(M; e) = U(x(M), y(M), s(M); e).$$

The interesting issue at this point is how the kink in the derived utility function of Section 3 is translated into this continuum setting. To understand this, consider rephrasing the optimization so that $x$ and $y$ are chosen conditional on $s$, giving a derived utility, and then $s$ is selected by the maximal derived utility. Graphically, there is a concave derived utility for each $s$ and the maximum value function is the upper envelope of these; see Figure 3. The analogue of the kink in this context is convexity of the maximum value function.

![Figure 3: Derived utility and maximum value function](image-url)
As can be appreciated from this construction, few restrictions are placed on the maximum value function except that it be strictly monotonically increasing and must be strictly concave for incomes below that where lowest quality is purchased and above that where highest quality is purchased. In between these limits, the second derivative is not restricted a priori.

An insight into the conditions that can lead to convexity can be easily obtained if the utility function is specialized to

\[ U = u_1(x) + u_2(y) + u_3(s) + u_4(e). \]

The following lemma can then be proved.

**Lemma 4.** For \( s(M) \in (\bar{s}, \bar{s}) \), \( V_{MM}(M;e) > 0 \) if

\[
\frac{u_3''}{u_3'} - \frac{q''r' + r'y}{q'r' + r'y} > 0. \tag{51}
\]

**Proof.** See Appendix. ||

The interpretation of Lemma 4 is that the second derivative of the maximum value function is positive (so the marginal utility of income increases with income) when the total cost of travelling distance \( y \) with a car of quality \( s \) is more concave then the utility of quality. Expressed intuitively, the condition requires that it is possible to move up the quality scale sufficiently fast to more than offset the decreasing marginal utility of consumption.

Consequently, although the continuum model does not have the kink involved in the discrete model, it can possess an increasing marginal utility of income. When it does the marginal utility effect of taxation will then point in the direction of subsidizing higher quality since this has the effect of providing welfare-enhancing transfers of income to the rich.

The utility function in (51) can also be used to investigate the externality effect of taxation. Referring back to the discussion of Section 4, what is of interest is the term \( \frac{\partial}{\partial M} e(s(M))y(M) \). The mileage reduction condition is equivalent to this derivative being negative, or

\[ e's'y + ey' < 0. \tag{52} \]

Since \( e' > 0 \) and \( s' > 0 \), it is obviously necessary that \( y' < 0 \), or that higher-income consumers travel less but substitute for this through higher quality.

Section 4 demonstrated that the nature of taxation, i.e. the relative taxation of high and low quality cars, was dependent upon the marginal utility and externality effects. The first of these could work towards the subsidization of high quality
because of the upward discontinuity in marginal utility. The equivalent of this in the continuum case is an increasing marginal utility of income. As shown by Lemma 4, the conditions necessary for this to hold are not strong. In addition, the mileage reduction condition can also hold. Putting these factors together, there are factors in the continuum case which can point to the subsidization of higher quality cars relative to lower quality.

7. Conclusions:
Private cars are significant contributors to air pollution. This has lead to a number of policy proposals for controlling their use. Amongst these has been the recommendation that existing taxes be adjusted to take account of environmental impacts. Since larger cars are less efficient in their use of fuel and are purchased by richer consumers, it would seem that the argument that they be taxed more heavily is entirely persuasive. One means to achieve this is to differentiate the annual tax in favour of small cars.

The paper constructed a model of vertical product differentiation in order to address these policy issues. When there is a discrete set of different qualities, it was shown that upward discontinuities are introduced into the marginal utility of income at the points where the quality of car chosen changed. These can reverse arguments concerning the redistributive role of taxation. This could occur to such an extent that high-quality cars should be subsidized relative to low quality. Furthermore, the additional expense of a higher-quality car means that its marginal purchaser will use it less than the marginal purchaser of low quality. If this results in a lower output of the externality, an additional reason is created for subsidizing high quality. The arguments were shown to extend to a continuum of qualities where the kink could be replaced by utility being convex in income.

These findings suggest that the policy proposals for discriminatory taxes on higher-quality cars should be treated cautiously. Vertical product differentiation introduces factors into the analysis which are not present in its absence. These can easily overturn standard reasoning and generate counter-intuitive conclusions.

There are a number of issues that arise when the practical interpretation of these results is considered. Foremost amongst these is the measurement of $s$. A number of the results are dependent upon the degree of concavity of price as a function of quality but this pre-supposes some cardinal measure of quality. Such a measure is not naturally available. This is less of a problem for the discrete quality model than it is for the continuum model, but it is one that arises in both. Any selection of a measure for quality (such as choosing it to make price linear) will have to be reflected in a equivalent selection of utility.

Appendix

Proof of Lemma 4.
The maximum value function is defined by
\[ V(M; e) = u_1(x(M)) + u_2(y(M)) + u_3(s(M)) + u_4(e). \]  

(A1)

Using the Envelope Theorem and the fact that \( x = M - q - ry \)

\[ V_M = u_1'. \]  

(A2)

so

\[ V_{MM} = u_1''[1 - (q' + r'y)s' - ry']. \]  

(A3)

Hence

\[ V_{MM} > 0 \iff 1 - (q' + r'y)s' - ry' < 0. \]  

(A4)

Using the first-order conditions for choice, the values of \( s' \) and \( y' \) are derived from the system

\[
\begin{bmatrix}
  u_1''[q' + r'y]^2 - u_1'[q'' + r''y] + u_2'' + u_1''r[q' + r'y] - u_1'r \\
  u_1''r[q' + r'y] - u_1'r' + u_1'r^2 + u_3''
\end{bmatrix}
\begin{bmatrix}
  ds \\
  dy
\end{bmatrix}
= \begin{bmatrix}
  u_1''[q' + r'y]dM \\
  u_1''rdM
\end{bmatrix}. \]  

(A5)

Substituting the solutions from (A5) into (A4) provides the condition in the statement of the lemma. ||

References


CSO, (1989), Social Trends.


