AN AGGREGATION THEOREM FOR THE VALUATION OF EQUITY UNDER LINEAR INFORMATION DYNAMICS

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ABSTRACT

We state an Aggregation Theorem which shows that the recursion value of equity is functionally proportional to its adaptation value. Since it is well known that under its existing business technology the recursion value of equity is equal to its book value plus the expected present value of its abnormal earnings, this implies that the adaptation value of equity can normally be determined by a process of simple quadrature. We demonstrate the application of the Aggregation Theorem using two stochastic processes. The first uses the linear information dynamics of the Ohlson (1995) model which is founded on the Uhlenbeck and Ornstein (1930) process. The second uses linear information dynamics based on the Cox, Ingersoll and Ross (1985) “square root” process. Both these processes lead to closed form expressions for both the adaptation and overall market values of equity. There are, however, many other processes which are compatible with the Aggregation Theorem. These all show that the market value of equity will be a highly convex function of its recursion value. The empirical evidence we report for U.K. companies largely supports the convexity hypothesis.

KEY WORDS: adaptation value, book to market ratio, earnings to book ratio, linear information dynamics, recursion value

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1. Introduction

The market value of equity is potentially, a complex non-linear function of a variety of “information” variables, including a firm’s earnings, the book values appearing on its balance sheet and perhaps, other contextual and economic variables as well. The literature recognises this relationship and probably the most exhaustive piece of empirical work published in the area is that of Burgstahler and Dichev (1997) who employ a North American sample of over 45,000 firm-years and from which they conclude that the market value of equity is a highly convex function of a firm’s earnings and the book value of its equity.

We can view the market value of a firm’s equity as being the outcome of taking a weighted average of its earnings and the book value of its equity, “the [weight or] coefficient associated with earnings increases with the ratio of earnings to book value and the [weight or] coefficient on book value decreases with the ratio of earnings to book value ....” [Burgstahler and Dichev (1997, p. 212)]. Burgstahler and Dichev (1997) conclude that this convex relationship arises out of the fact that a firm can be viewed as a set or collection of resources to which it applies a particular “business technology” (or way of combining and using its resources) to produce a stream of expected future earnings. There are, as a consequence, two complementary aspects to the valuation of a firm’s equity. The first of these is determined by discounting the stream of expected future earnings under the assumption that the firm applies its existing business technology indefinitely into the future. This is defined as the “recursion value” of equity. The second element of value arises out of the fact that the firm invariably has options to “convert” or use its resources in alternative and potentially more profitable ways; that is, the firm has options to change its existing business technology. Firms can exercise these options by employing “liquidations, sell-offs, spin-offs, divestitures, CEO changes, mergers, takeovers, bankruptcies, restructurings, and new capital investments” as vehicles through which to change the way they use their existing resources. The potential to make changes like these gives rise to what is known as the “adaptation value” of equity [Burgstahler and Dichev (1997, p. 188)].

Now, it is well known that if the firm applies its existing business technology indefinitely into the future that the recursion value of equity is equal to the current book value of
equity plus the expected present value of its future abnormal earnings [Edey (1962, pp. 201-202), Ohlson (1995, pp. 666-667)]. This also means that if the firm’s earnings evolve as a Markov process under its existing business technology\(^2\), then the recursion value of equity can be determined in terms of the current book value of equity and its current abnormal earnings [Ohlson (1995, pp. 668-669)]. However, the calculation of adaptation value has always been more troublesome, and attempts at determining an analytical expression for it, even for standard stochastic processes, have proved to be largely fruitless [Burgstahler and Dichev (1997, p. 194), Yee (2000)]. Thus, our objective here, and the principal contribution of our paper, is to employ a generalisation of the linear information dynamics which underpin the Ohlson (1995) valuation model to present an Aggregation Theorem which shows that the recursion value of equity is functionally proportional to its adaptation value. This has the important implication that once the recursion value of equity is known, it will normally be a relatively simple exercise to determine the adaptation value of equity. And since the market value of equity is the sum of its recursion value and its adaptation value, the exact nature of the convex relationship between the market value of equity on the one hand and current earnings and the book value of equity on the other, can also be determined in a relatively straightforward manner.

In the next Section we give a graphical account of the relationship between the recursion and adaptation values of equity and the overall market value of equity. We then state our Aggregation Theorem and use the Ohlson (1995) linear information dynamics to demonstrate how the Theorem may be applied in practice. The generalised nature of the Theorem is then demonstrated by determining the adaptation and recursion values of equity when the linear information dynamics depend upon what we regard as a more realistic “square root” error structure. The penultimate section summarises empirical evidence from the United Kingdom which is largely compatible with the kind of non-linearities in equity value suggested by our analytical results. Our paper concludes with some brief summary comments.

\(^2\) Here it is important to note that it is not unreasonable to assume that earnings evolve in terms of a Markov process since “… most univariate time series research shows that earnings, on average, follow a random walk. In fact, Bernard (1994) finds that it is difficult to improve on current earnings as a predictor of future earnings.” [Burgstahler and Dichev (1997, p. 193)]
2. The Aggregation Theorem

Figure I demonstrates the nature of the relationship between the market value of a firm’s equity on the one hand and its recursion and adaptation values on the other. The downward sloping curve represents the adaptation value of equity. This captures the value of the options the firm has to use its resources in alternative and potentially more profitable ways or even, to terminate or shut down its existing operations. The upward sloping line emanating from the origin represents the recursion value of equity. The recursion value of equity is equal to the current book value of equity plus the expected present value of its future abnormal earnings, given that the firm applies its existing business technology indefinitely into the future.

The abnormal earnings are the difference between the book (or accounting) earnings and an imputed return to capital obtained by multiplying the cost of equity capital by the book value of equity. Note that when the expected present value of the future abnormal earnings is a large negative figure relative to the book value of equity, then the recursion value of equity will normally be very small. Equity value will then be principally composed of its adaptation value. This is reflected in the fact that near the origin, the recursion value of equity is trivially small. Thus, since the market value of equity is the sum of its recursion and adaptation values, then near the origin, the market value of equity is composed almost entirely of its adaptation value. When, however, the expected present value of future abnormal earnings is large relative to the book value of equity, then the market value of equity will normally be composed mainly of its recursion value. Thus, as we move further and further towards the right of the graph, the vertical distance between the market value of equity and its recursion value becomes trivially small. This means that equity value will be composed almost entirely of its recursion value. In
between these two extremes, however, the market value of equity will consist of a more balanced combination of its recursion and adaptation values.

Unfortunately, analytical developments in this area have tended to lag behind empirical applications [Berger, Ofek and Swary (1996, p. 261), Burgstahler and Dichev’s (1997, p. 212)]. In particular, the literature provides little guidance about exactly how one might go about decomposing the market value of a firm’s equity into its adaptation and recursion components. Thus, we now employ a generalisation of the linear information dynamics which underpin the Ohlson (1995) valuation model to present an Aggregation Theorem which shows that the recursion value of equity is functionally proportional to its adaptation value. This means that once the recursion value of equity is known, its adaptation value can be computed as an exercise in simple quadrature. The market value of equity is then easily decomposed into its recursion and adaptation components. The important definitions and assumptions on which the Aggregation Theorem is based are stated in parts (i) and (ii) of the Theorem; the conclusions to the Theorem are contained in parts (iii) and (iv). The Appendix contains an outline proof of the Theorem.

**Theorem**

(i) Let \( i \) be the cost of capital (per unit time) for the firm’s equity, \( b(t) \) be the book value of equity and \( a(t) \) be the instantaneous abnormal earnings (per unit time) attributable to equity, all at time \( t \). Define the recursion value of equity, \( \eta(t) \), as the book value of equity plus the expected present value of the future abnormal earnings stream, or:

\[
\eta(t) = b(t) + E_t\left[\int_t^\infty e^{-i(s-t)}a(s)ds\right]
\]

where \( E_t(\cdot) \) is the expectations operator taken at time \( t \), conditional on the firm applying its existing business technology indefinitely into the future.
(ii) Suppose that under its existing business technology the firm’s abnormal earnings, a(t), and an “information variable”, \( \nu(t) \), evolve in terms of the linear information dynamics:

\[
\begin{pmatrix}
da(t) \\
d\nu(t)
\end{pmatrix} = \begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix}\begin{pmatrix}
a(t) \\
\nu(t)
\end{pmatrix} dt + \eta \delta(t)\begin{pmatrix}
k_1dz_1(t) \\
k_2dz_2(t)
\end{pmatrix}
\]

where \( 0 \leq \delta \leq \frac{1}{2} \) is a real number, \( c_{11}, c_{12}, c_{21}, \) and \( c_{22} \) are parameters defining the “structural matrix”, \( k_1 = \frac{(i - c_{11})(i - c_{22}) - c_{21}c_{12}}{(i - c_{22})} \) and \( k_2 = \frac{(i - c_{11})(i - c_{22}) - c_{21}c_{12}}{c_{12}} \) are “normalising” constants, and \( dz_1(t) \) and \( dz_2(t) \) are Wiener processes with variance parameters \( \sigma_1^2 \) and \( \sigma_2^2 \) and correlation parameter, \( \rho_{12} \).

(iii) It then follows that the recursion value of equity evolves in accordance with the stochastic differential equation:

\[
d\eta(t) = i\eta(t)dt + \eta \delta(t)dq(t)
\]

where \( dq(t) = (dz_1(t) + dz_2(t)) \) is a Wiener process with variance parameter \( \sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2 \). Furthermore, the adaptation value of equity is:

\[
\kappa(\eta) = h\eta \int_0^{\frac{1}{\eta}} \exp\left[-\frac{\theta z^2(\delta - 1)}{2(1 - \delta)}\right]dz
\]

where \( \theta = \frac{2i}{\sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2} > 0 \) and \( h \) is a “normalising” constant.

(iv) Finally, the market value of equity, \( P(\eta) \), will be the sum of its recursion and adaptation values, or:

\[
P(\eta) = \eta + \kappa(\eta)
\]

3 The condition, \( 0 \leq \delta \leq \frac{1}{2} \), given in part (ii) of the Aggregation Theorem ensures that the expression for the recursion value of equity given in part (iii) and the overall value of equity given in part (iv) of the theorem, both converge for all values of \( \eta \geq 0 \). We can demonstrate this by letting \( \delta = 1 \) in which case the recursion value of equity evolves in terms of the stochastic differential equation:

\[
d\eta(t) = i\eta(t)dt + \eta(t)dq(t)
\]

This is the well known equation for the geometric Brownian motion which underscores the option pricing formula of Black and Scholes (1973). Following the procedures contained in the Appendix shows that if the
\[
P(\eta) = \eta + h\eta \int_{0}^{1} \exp[-(\theta z^2(\delta-1)/(2(1-\delta))]dz
\]

The first observation we need to make about this Theorem is that although the linear information dynamics are restricted to just two stochastic variables (residual income and the information variable), the Theorem generalises to any finite number of variables. Thus, one could include other variables besides these two, and as long as this augmented system of variables evolves in terms of the linear information dynamics with the same error structure as that contained in part (ii) of the Aggregation Theorem, there will be no difference in the conclusions reached in parts (iii) and (iv) of the Theorem.\(^4\)

Second, previous work in the area has invariably included the “information variable”, \(\nu(t)\), as part of the information dynamics but seldom is any explanation given as to the recursion value of equity does evolve in terms of a geometric Brownian motion, then the market value of equity, \(P(\eta)\), will have to satisfy an Euler equation whose solution is [O’Neil (1987, pp. 138-139)]:

\[
P(\eta) = \eta + h\eta^{\theta}
\]

where \(\theta\) and \(h\) are defined in part (iii) of the Theorem. This solution takes the form of an unbounded “cusp" (Limit \(P(\eta) \to \infty\) and Limit \(P(\eta) \to -\infty\) with a minimum at the point \(\eta = (\theta h)^{\frac{1}{\theta+1}}\). Since we do not observe firms whose equity value is compatible with this pricing relationship, we do not consider it further here.

\(^4\) Consider the more general linear information dynamics:

\[
\begin{pmatrix}
dv_{1}(t) \\
dv_{2}(t) \\
\vdots \\
dv_{n-1}(t) \\
dc(t)
\end{pmatrix} =
\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1n} \\
c_{21} & c_{22} & \cdots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \cdots & c_{nn}
\end{pmatrix}
\begin{pmatrix}
v_{1}(t) \\
v_{2}(t) \\
\vdots \\
v_{n-1}(t) \\
av(t)
\end{pmatrix}
+ \begin{pmatrix}
k_{1} dz_{1}(t) \\
k_{2} dz_{2}(t) \\
\vdots \\
k_{n} dz_{n}(t)
\end{pmatrix}
\]

based on the residual income variable, \(a(t)\), and the “information” variables, \(v_{j}(t), j = 1, 2, \ldots, n-1\). The \(v_{j}(t)\) might, for example, be components of the residual income variable, \(a(t)\), such as accruals, prepayments, depreciation expense, research and development expenditure, book value of fixed assets, sales, etc., [Lev (2001), Barth, Cram and Nelson (2001)]. Alternatively, they might be components of the book value, \(b(t)\), and are thus used to capture how “specialised” the firm’s assets might be [Berger, Ofek and Swary (1996)]. Furthermore, since there is growing evidence to suggest that there are occasions when a firm’s bookkeeping summary measures are largely value-irrelevant, these information variables might also include factors not (immediately) captured by the bookkeeping system [Amir and Lev (1996)]. The important observation to be made here is that our analysis shows equity prices to be a potentially non-linear function of all these variables - in contrast to the mainly linear technologies which characterise most of the empirical research reported in this area of the literature.

\[^4\]
“meaning” of this variable. Using the statement of the linear information dynamics given in part (ii) of the Aggregation theorem shows, however, that the abnormal earnings evolve in accordance with the process:

\[
da(t) = (c_{11}a(t) + c_{12}\nu(t))dt + k_1\eta(t)dz_1(t)
\]
or:

\[
da(t) = -c_{11}\left(\frac{-c_{12}}{c_{11}}\nu(t) - a(t)\right)dt + k_1\eta(t)dz_1(t)
\]

Thus, if we assume \(c_{11} < 0\) and \(c_{12} > 0\), then it follows that the abnormal earnings are generated by a mean reversion process with a speed of adjustment coefficient given by \(c_{11}\). The speed of adjustment coefficient is a measure of the strength with which the current abnormal earnings, \(a(t)\), are drawn back towards their current “long run” normal value, which is \(-\frac{c_{12}}{c_{11}}\nu(t)\). It thus follows that the information variable, \(\nu(t)\), is nothing more than an index for, or a “scaled” value of, the long run abnormal earnings towards which the current abnormal earnings are currently evolving.

Note also, that part (iii) of the theorem shows that the recursion value of equity has an expected instantaneous growth rate (per unit time), \(\frac{1}{\eta(t)}E_t[d\eta(t)]dt = i\), equal to the cost of capital for the equity security. However, whilst our expectation will be for the recursion value of equity to grow at this rate, it will do so stochastically with a variance (per unit time) of \(\text{Var}_t\left[\frac{d\eta(t)}{\eta(t)}\right] = (\sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2)\eta^{2(\delta-1)}(t)\), where \(\text{Var}_t(\cdot)\) is the variance operator taken at time \(t\). Furthermore, since the growth rate in recursion value is normally distributed, then the distributional properties of the growth rate in recursion value will be completely known once we know its mean and variance.

\(^5\) It is not hard to show that the expected instantaneous growth rate in adaptation value will also be equal to the cost of equity capital, or \(\frac{1}{\kappa(\eta)}E_t[d\kappa(\eta)]dt = i\).
It is also important to note that part (iii) of the theorem shows 
\[ \theta = \frac{2i}{\sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2} \] 
to be a crucial parameter in determining the relative magnitude 
of an equity security’s adaptation value. This variable is (twice) the ratio of the cost of 
equity capital, i, divided by the variance parameter, \( \frac{\sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2} \), of the Wiener 
process (dq(t)) on which the variance of instantaneous changes in the equity security’s 
recursion value depends. As such, \( \theta \) is a measure of the relative stability of the 
instantaneous return that accrues on the equity security’s recursion value. Now, as the 
value of the variance parameter increases relative to the cost of equity, then \( \theta \) declines 
and \( \exp[-\frac{\theta z^2(\delta-1)}{2(1-\delta)}] \) grows in value and so, the adaptation value of equity increases. This 
in turn means that the adaptation value of equity increases as the variability of its 
recursion value increases. Similarly, the adaptation value of equity decreases as the 
variability of its recursion value decreases. This is as we would expect it to be. For when 
steady growth in recursion value in line with the cost of equity is assured, it is unlikely 
that the catastrophic events which will induce the firm to exercise its adaptation options, 
will arise. Hence, the small probability of these options ever being exercised will mean 
that the adaptation value implied by them, will also have to be small.

Finally, the relationship between the recursion value of equity on the one hand 
and the book value of equity, residual income and the information variable on the other, 
is summarised in the following Lemma:

**Lemma**

Under the definitions and assumptions contained in parts (i) and (ii) of the above 
Aggregation Theorem, the recursion value of equity is:

\[ \eta(t) = b(t) + \frac{(i - c_{22})a(t)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} + \frac{c_{12}v(t)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} \]
A simple application of the Aggregation Theorem is provided by the Ohlson (1995) system of linear information dynamics which uses the (discrete time interpretation of the) Uhlenbeck and Ornstein (1930) process to model the stochastic relationship between earnings, the information variable and the book value of equity. This leads to a system of linear information dynamics based on the assumed parameters $c_{11} = -\beta$, $c_{12} = 1$, $c_{21} = 0$, $c_{22} = -\gamma$, $\delta = 0$, $k_1 = (\beta + i)$ and $k_2 = (i + \beta)(i + \gamma)$. Substituting these values into the above Lemma shows that the recursion value of equity under the Ohlson (1995) linear information dynamics is:

\[
\eta(t) = b(t) + \frac{a(t)}{(1 + \beta)} + \frac{\nu(t)}{(i + \beta)(i + \gamma)}
\]

Furthermore, substitution into part (iii) of the Aggregation Theorem shows that the adaptation value of equity under the Ohlson (1995) linear information dynamics will be:

\[
\kappa(\eta) = \frac{1}{h} \int_{0}^{1} \exp\left(-\frac{\theta z^2}{2}\right) dz
\]

Now, here it can be shown Limit $\eta \to 0$ $\kappa(\eta) = h \geq 0$, or that when the recursion value of equity, $\eta$, is relatively “small”, the adaptation value of equity, $\kappa(\eta)$, is relatively “large”.6 Furthermore, since Limit $\eta \to \infty$ $\kappa(\eta) = 0$, it also follows that the recursion value of equity will

6 Defining $g\left(\frac{1}{\eta}\right) = \int_{0}^{1} \exp\left(-\frac{\theta z^2}{2}\right) dz$ means that the recursion value of equity can be stated as $\kappa(\eta) = \frac{1}{h} g\left(\frac{1}{\eta}\right)$. Now, Limit $g\left(\frac{1}{\eta}\right)$ is non-convergent [Spiegel (1974, p. 264)] and so, Limit $\kappa(\eta)$ takes the indeterminate form $\frac{\infty}{\infty}$. However, applying L’Hôpital’s Rule [Spiegel (1974, p. 62)] shows

Limit $\kappa(\eta) = \text{Limit } h g'\left(\frac{1}{\eta}\right) = \text{Limit } h \exp\left(-\frac{\theta \eta^2}{2}\right) = h$. 

Limit $\eta \to 0$ $\eta \to 0$ $\eta \to 0$
be relatively “large” when its adaptation value is relatively “small”. These results imply that when the expected present value of future abnormal earnings is low, most of the market value of equity is attributable to its adaptation value and a revision in recursion value matters little for the valuation of equity. However, when the expected present value of future abnormal earnings is large, most of the market value of equity is attributable to its recursion value and thus, a revision of this recursion value has a large effect on the market value of equity. For intermediate levels of recursion value, the market value of equity depends on a more balanced combination of its adaptation and recursion values.

We can now use part (iv) of the Aggregation Theorem to bring the recursion and adaptation values together to state the overall market value of equity as:

\[
P(\eta) = \eta + P(0)\eta \int_{0}^{1} \exp(-\theta z^2)dz
\]

It is easily shown that this valuation formula is consistent with the convex stochastic relationship between earnings and the market value of equity documented in the North American empirical literature [Bernard (1994), Burgstahler and Dichev (1997), Penman (1996, 1998)]. Before demonstrating this, however, we illustrate the generalised nature of the Aggregation Theorem by determining the adaptation and recursion values of equity when there is “biased” accounting and the linear information dynamics depend upon a more realistic “square root” error structure.

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7 In this case, \( \lim_{\eta \to \infty} \kappa(\eta) \) takes the indeterminate form \( \frac{0}{0} \). However, we can again apply L’Hôpital’s Rule to show that \( \lim_{\eta \to \infty} \kappa(\eta) = \lim_{\eta \to \infty} h.g'(\frac{1}{\eta}) = \lim_{\eta \to \infty} h.exp(-\frac{\theta \eta^2}{2}) = 0. \)

8 Taking limits shows \( \lim_{\eta \to 0} P(\eta) = \lim_{\eta \to 0} \kappa(\eta) = h \), or that \( h = P(0) \).

9 It bears emphasising, however, that the functional form of the relationship between earnings, book value and the information variable on the one hand and equity value on the other, is markedly different from the empirically based specifications one finds in the literature of the area [Burgstahler and Dichev (1997), Penman (1998), Hand (2000), Eastman, Taylor, Shroff and Sougiannis (2000)].
3. “Biased” Accounting and “Square Root” Linear Information Dynamics

Whilst the Ohlson (1995) linear information dynamics are well known and therefore represents a useful and reasonably parsimonious interpretation of our Aggregation Theorem, these linear information dynamics are unrealistic in the sense that they imply that increments in the recursion value of equity do not depend on the existing value of this variable. This means that the variance of instantaneous changes in the recursion value of equity will be the same, irrespective of whether the recursion value is currently £1 or £1 billion. There is, however, ample evidence to suggest that fluctuations in economic time series become more volatile as the affected variable assumes more “extreme” values [Cox, Ingersoll and Ross (1985)]. Furthermore, the Ohlson (1995) model assumes “unbiased” accounting in the sense that the index of abnormal profits, \( \nu(t) \), perpetually gravitates towards a long run mean of zero. This assumption could probably be justified in an era when “… companies achieved competitive advantage from their investment in and management of tangible assets such as inventory, property, plant and equipment …”, all of which normally receive formal expression on corporate balance sheets. However, “… by the end of the 20th century, intangible assets became the major source for competitive advantage. In 1982, tangible book values represented 62 percent of industrial organisations’ market values; ten years later, the ratio had plummeted to 38 percent …. By the end of the 20th century, the book value of tangible assets accounted for less than 20 percent of companies’ market values ….” [Kaplan and Norton (2001, p. 88)]. Statistics like these mean that balance sheets have increasingly failed to reflect the resources on which corporate activities are based and so, the capital charge on which the residual income measure has been traditionally based is likely to be understated. It thus follows that residual income measures based on the traditional balance sheet will be overstated and is unlikely to gravitate towards a mean of zero over the long run [Pope and Walker (1999)]. Hence, it is doubtful if the “unbiased” accounting assumption, on which most of the models in this area are founded, will hold up in practice.

We seek to address these issues by stating a second and more general model which makes provision for the fact that a firm’s accounting is likely to be biased and also, that the uncertainty associated with the evolution of earnings and book values will increase as these variables become larger. We thus assume \( c_{12} = -c_{11} \) and \( \delta = \frac{1}{2} \) in part
(ii) of our Aggregation Theorem, in which case the residual income variable evolves in terms of the following stochastic differential equation:

$$da(t) = c_{12}(ν(t) - a(t))dt + k_1\sqrt{η(t)}.dz_1(t)$$

where $$k_1 = \frac{(i + c_{12})(i - c_{22}) - c_{21}c_{12}}{(i - c_{22})}$$, $$c_{21} > 0$$ and $$c_{22} < 0$$. Note that this assumption implies that the residual income variable gravitates towards a long term mean of $$ν(t)$$ with a restoring force which is proportional to the difference between the current residual income and its long term mean. The constant of proportionality is given by the speed of adjustment coefficient, which in this instance is, $$c_{12} = -c_{11}$$. Furthermore, the variance (per unit time) of instantaneous changes in the residual income variable is $$k_1^2\sigma_1^2$$, a variable which grows in line with increases in the firm’s recursion value, $$η(t)$$. Hence, as the firm grows, so does the uncertainty associated with its earnings and book values. Furthermore, these assumptions will also imply that the long run mean of the residual income variable evolves in accordance with the following process:

$$dν(t) = (c_{21}a(t) + c_{22}ν(t))dt + k_2\sqrt{η(t)}.dz_2(t)$$

where $$k_2 = \frac{(i + c_{12})(i - c_{22}) - c_{21}c_{12}}{c_{12}}$$. This process encapsulates the assumption that the residual income variable will not, in general, gravitate towards a long run mean of zero but rather will tend to grow in both size and uncertainty over time. This captures the fact that both the firm’s balance sheet and its residual income measure, fail to reflect the totality of the resources on which its production, investment and financing activities are based.\(^\text{10}\)

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\(^{10}\) Whether the residual income variable, $$a(t)$$, converges towards zero (a necessary condition for unbiased accounting) hinges on the eigenvalues of the structural matrix, $$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$, given in part (ii) of the Aggregation Theorem [O’Neil (1987, pp. 373-386), Ashton (1997), Tippett and Warnock (1997, pp. 1076-1084)]. For the particular interpretation of the structural matrix taken here ($$c_{12} = -c_{11}$$), it may be shown that if $$c_{21} > -c_{22}$$, then the residual income variable will have explosive properties and will not, therefore, converge towards zero. In other words, if this condition is satisfied then the accounting will, in general, be biased.
Substituting the parameters implied by these stochastic processes into the Lemma which returns the recursion value of equity, shows:

\[ \eta(t) = b(t) + \frac{(i - c_{22})a(t)}{(i + c_{12})(i - c_{22}) - c_{21}c_{12}} + \frac{c_{12}v(t)}{(i + c_{12})(i - c_{22}) - c_{21}c_{12}} \]

to be the recursion value of equity under the biased accounting assumptions implied by the square root linear information dynamics. Furthermore, part (iii) of the Aggregation Theorem shows that the adaptation value of equity under the square root information dynamics will be:

\[ \frac{1}{\eta} \kappa(\eta) = h \int_{0}^{1} \eta \exp(-\theta z^{-1}) dz \]

Now, for this model too, we have that \( \lim_{\eta \to 0} \kappa(\eta) = h \geq 0 \), or that when the recursion value of equity, \( \eta \), is relatively “small”, the adaptation value of equity, \( \kappa(\eta) \), is relatively “large”. Furthermore, since \( \lim_{\eta \to \infty} \kappa(\eta) = 0 \), it also follows that the recursion value of equity will be relatively “large” when its adaptation value is relatively “small”. This means that we can also use part (iv) of the Aggregation Theorem to bring the recursion and adaptation values together into the following expression for the overall market value of equity:

\[ \frac{1}{\eta} \eta + \int_{0}^{1} \eta \exp(-\theta z^{-1}) dz \]

This square root linear information dynamics pricing model is based on the more realistic assumption that fluctuations in the recursion value of equity become more volatile as the recursion value itself becomes larger. It also assumes that the residual income variable will not necessarily gravitate towards a long run mean of zero. In other words, it does not impose the “unbiased” accounting assumptions which one frequently encounters in this area of the literature. Finally, the square root pricing model is also consistent with the convex stochastic relationship between earnings and the market value of equity.
documented in the empirical literature, something which we now demonstrate in greater detail.

4. Earnings and the Value of Equity

Previous analysis shows that both the Ohlson (1995) and “square root” linear information dynamics lead to a model for which the recursion value of equity will assume the general form:

\[ \eta(t) = b(t) + \frac{(i - c_{22})a(t)}{(i - c_{12})(i - c_{22}) - c_{21}c_{12}} + \frac{c_{12}v(t)}{(i - c_{12})(i - c_{22}) - c_{21}c_{12}} \]

Here, it will be recalled that the Ohlson (1995) model is based on the interpretation of this formula which assumes \( c_{11} = -\beta, c_{12} = 1, c_{21} = 0, c_{22} = -\gamma, \delta = 0, k_1 = (\beta + i) \) and \( k_2 = (i + \beta)(i + \gamma) \). The square root model, however, is based on the less restrictive interpretation which assumes \( c_{11} = -\beta, c_{12} = -c_{11}, c_{21}, c_{22} > 0, \delta = \frac{1}{2} \), \( k_1 = \frac{(i + c_{12})(i - c_{22}) - c_{21}c_{12}}{(i - c_{22})} \) and \( k_2 = \frac{(i + c_{12})(i - c_{22}) - c_{21}c_{12}}{c_{12}} \). Now, under the clean surplus identity, book or accounting earnings are related to the book value of equity and abnormal earnings through the formula, \( x(t) = a(t) + ib(t) \), where \( x(t) \) is the instantaneous accounting earnings, \( i \) is the cost of equity capital, \( a(t) \) is the instantaneous abnormal earnings and \( b(t) \) is the book value of equity at time \( t \). Using this definition and a little algebra shows that the recursion value of equity can then be restated in terms of the book earnings attributable to equity, as follows:

\[ \eta(t) = \frac{(i - c_{22})x(t) - (c_{11}(i - c_{22}) + c_{21}c_{12})b(t) + c_{12}v(t)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} \]

Now, minimal regularity conditions show that if we take the derivative of the price of the equity security, \( P(\eta) \), with respect to book (or accounting) earnings, \( x \), we have:

\[ \frac{\partial P}{\partial x} = \frac{\partial \eta}{\partial x} \frac{dP}{d\eta} = \frac{c_{12}v(t)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} \frac{dP}{d\eta} > 0 \]

and:
\[
\frac{\partial^2 P}{\partial x^2} = \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{d^2 P}{d \eta^2} = \frac{(i - c_{22})^2}{[(i - c_{11})(i - c_{22}) - c_{21}c_{12}]^2} \frac{d^2 P}{d \eta^2} > 0
\]

These are necessary conditions for a convex relationship to exist between the value of equity on the one hand and the book or accounting earnings attributable to equity on the other. Thus, our analysis demonstrates that there are persuasive analytical reasons for believing there ought to be a convex relationship between the value of equity, \( P(\eta) \), and accounting earnings, \( x \).

Burgstahler and Dichev (1997) demonstrate that the convex relationship predicted by our analysis is largely confirmed by the North American data which they investigate. There are, however, areas of significant difference between North American accounting standards and practices and those which prevail in other advanced industrialised countries. Weetman, Jones, Adams and Gray (1998), for example, show that there are significant differences between the North American and United Kingdom accounting standards in the treatment of goodwill, pension costs and deferred taxation amongst other items. Differences in accounting standards could lead to different empirical regularities across national boundaries and so, we now report the results of our investigation into the relationship between the market value of equity and accounting earnings for a large sample of U.K. companies.

We began our analysis by computing the ratio of annual earnings to the book value of equity for the eleven year period from 1988 until 1998 for all U.K. firms available on the Datastream system over this period.\(^{11}\) Earnings were defined as “earned for ordinary”, which is Datastream item #625. Book value was defined as “net tangible assets” at the beginning of the year over which the earnings are computed, which is Datastream item #305 less item #344.\(^{12}\) We also computed the ratio of the market value of equity to the

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\(^{12}\) Burgstahler and Dichev (1997, p. 195) choose opening book value as the measure of adaptation value for the year because by definition, closing book value includes the current year’s earnings as a component. Therefore, empirical analysis using opening book values will separate the effects of earnings and book values more sharply. It is also worth emphasising that basing our empirical analysis on alternative measures of book value (e.g., tangible assets, Datastream item #305) does not result in any qualitative differences when compared to the results we report in the text.
book value of equity, for this sample of companies. The market value of equity was based on the share price prevailing 100 days after the (end of year) balance sheet date. Finally, we computed the market to book ratio based on the share price 40 days and 160 days after the balance sheet date. Since there were no qualitative differences between the relationship of the market to book ratio and the earnings to book ratio at the three dates, only results based on share prices prevailing 100 days after the balance sheet date are summarised here. Summary results from this exercise are contained in Table I.

The first column in this Table gives the year relating to the summary statistics. The next three columns are the simple average, standard deviation and median of the observations for that year. Thus, the sample statistics for 1990 are based on 1051 firms and return a simple arithmetic average earnings to book ratio of 0.1544 across these firms. The standard deviation of the 1051 firms comprising this sample is 0.1492. Finally, the median observation for the sample is 0.1454. The next three columns give equivalent statistics for the market to book ratio of equity whilst the final column gives the number of firms for which there are negative earnings in that year. Thus in 1990, 84 firms returned negative earnings and 1051 - 84 = 967 firms returned positive earnings.

A plot of the relationship between the market value of equity and earnings, both scaled by book value, for the entire 12,547 observations on which Table I is based, is contained in Figure II. Note that this graph is compatible with the equivalent graph for North American firms contained in Burgstahler and Dichev (1997, p. 199). Thus, at “low” earnings levels, the relationship between the market value of equity and the earnings attributable to equity is fairly flat. However, as earnings increase the slope of the
relationship also increases reflecting the fact that over most of its domain, the relationship between the market value of equity and earnings appears to be one of convexity. These are empirical results which are largely compatible with the convex relationship our analysis suggests ought to exist between the market value of equity and book earnings.

At very high and low levels of earnings, however, there is evidence of an attenuation effect first suggested by Miller (1994) and found to be compatible with at least some North American data by Burgstahler (1998). At significantly large profitability levels Burgstahler (1998, pp. 336-337) argues that “abnormal earnings tend to be dissipated by economic forces” as a result of “competition or … political or regulatory actions.” Similarly, at low levels of earnings competition tends to drive firms out of business thereby restoring profitability to more reasonable levels for the firms which continue to operate. To see if there is evidence that U.K. firms are also affected by these phenomena, Figure III summarises results from ordering our data, based on the ratio of earnings to the book value of equity, into twenty sub-groups containing approximately 600 observations each and then, computing the simple average of the market to book ratios of equity for each sub-group. Again, this graph shares many of the properties of its North American counterpart, as summarised in the graphs prepared by Burgstahler (1998, p. 339). At “low” earnings levels there appears to be a concave relationship between earnings and the market value of equity. However, as earnings grow, this relationship turns to one of convexity. Finally, for the highest sub-groups there is evidence that the relationship between market value and earnings has again turned to one of concavity, although the evidence does not appear to be as “clear cut” as that produced by
Burgstahler (1998, p. 339) for the North American data on which his graphs are based. Thus, whilst our empirical evidence for the U.K. is broadly supportive of our hypothesis that there ought to be a convex relationship between the market value of equity and earnings, there is also evidence that at extremely low and high levels of earnings the relationship turns to one of concavity. This is compatible with an attenuation effect caused by firms either entering (or leaving) the relevant industry due to the rents that are being earned.\textsuperscript{13}

5. Summary Conclusions

The literature suggests that the market value of equity is potentially, a complex non-linear function of a variety of “information” variables, including a firm’s earnings, the book values appearing on its balance sheet and perhaps, other contextual and economic variables as well. However, analytical developments in this area have lagged far behind the burgeoning volume of empirical work, so much so that there are strong suspicions in many quarters that the empirical results appearing in the literature are based on inappropriate econometric methodologies [Burgstahler (1998)]. Hence, we respond here to Burgstahler and Dichev’s (1997, p. 212) call for the development of more refined valuation models by formulating a set of procedures which enable one to determine the value of a firm’s equity when all relevant “information variables” evolve stochastically and continuously through time. It turns out that our modelling procedures return a closed form solution for the value of equity and that this is a highly non-linear function of equity’s recursion and adaptation values.

Our principal contribution lies in stating an Aggregation Theorem which shows that the recursion value of equity is functionally proportional to its adaptation value. Now, it is well known that the recursion value of equity is equal to its book value plus the expected present value of its abnormal earnings, given that the firm applies its existing business technology indefinitely into the future. Thus, once the stochastic processes on which

\textsuperscript{13} An alternative explanation for this concavity phenomenon in the lowest sub-groups of the earnings to book ratio is provided by Hand (2000). He argues that accounting standards like the United Kingdom’s FRS10, “Goodwill and Intangible Assets”, typically require that “unprofitable” firms prematurely expense their large investments in goodwill and other intangible assets. However, the capital market “unravels” the accounting conservatism implied by these standards and it is this which leads to the concave relationship between the market to book and earnings to book ratios for highly unprofitable firms. He finds broad support for this hypothesis for the large sample of “pure play” internet firms which form the basis of his study. Lev (2001) summarises the accounting and disclosure requirements in this area for most of the industrialised nations.
earnings depend are identified, it is normally a simple matter to compute the recursion value of equity. Furthermore, the adaptation value of equity can then be determined by simple quadrature using the computation procedures laid down in part (iii) of the Aggregation Theorem. In the worst case scenario this means that the adaptation value of equity will have to be determined using numerical techniques. However, when earnings evolve in terms of some of the “standard” stochastic processes encountered in the literature, it will normally be possible for adaptation value to be expressed in closed form. Thus, both the Uhlenbeck and Ornstein (1930) process on which the Ohlson (1995) linear information dynamics is based and the Cox, Ingersoll and Ross (1985) “square root” process on which the “square root” linear information dynamics is founded, lead to closed form expressions for both the adaptation and overall market values of equity.
### TABLE I

**SUMMARY STATISTICS FOR BOOK (OR ACCOUNTING) EARNINGS DIVIDED BY THE BOOK VALUE OF EQUITY AND MARKET VALUE OF EQUITY DIVIDED BY THE BOOK VALUE OF EQUITY**

**N = 12,547 FIRM YEARS COVERING THE PERIOD FROM 1988 UNTIL 1998**

<table>
<thead>
<tr>
<th>Year</th>
<th>( \frac{E_t}{BV_{t-1}} ) Mean</th>
<th>( \frac{E_t}{BV_{t-1}} ) Std Dev</th>
<th>( \frac{E_t}{BV_{t-1}} ) Median</th>
<th>( \frac{MV_t}{BV_{t-1}} ) Mean</th>
<th>( \frac{MV_t}{BV_{t-1}} ) Std Dev</th>
<th>( \frac{MV_t}{BV_{t-1}} ) Median</th>
<th>Sample Size</th>
<th>Negative Earnings</th>
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<tr>
<td>87</td>
<td>0.1714</td>
<td>0.1248</td>
<td>0.1513</td>
<td>2.3005</td>
<td>1.4657</td>
<td>1.8706</td>
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<td>15</td>
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<td>88</td>
<td>0.1980</td>
<td>0.1225</td>
<td>0.1844</td>
<td>2.5997</td>
<td>1.7344</td>
<td>2.1114</td>
<td>1000</td>
<td>15</td>
</tr>
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<td>89</td>
<td>0.1865</td>
<td>0.1310</td>
<td>0.1683</td>
<td>2.3154</td>
<td>1.7062</td>
<td>1.8177</td>
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<td>32</td>
</tr>
<tr>
<td>90</td>
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<td>0.1492</td>
<td>0.1454</td>
<td>1.8589</td>
<td>1.4237</td>
<td>1.4382</td>
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<tr>
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<td>0.1029</td>
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<td>12547</td>
<td>1383</td>
</tr>
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</table>
FIGURE I

RELATIONSHIP BETWEEN ADAPTATION VALUE, RECURSION VALUE AND THE OVERALL MARKET VALUE OF EQUITY
FIGURE II

PLOT OF THE MARKET VALUE OF EQUITY DIVIDED BY THE BOOK VALUE OF EQUITY AGAINST BOOK (OR ACCOUNTING) EARNINGS DIVIDED BY THE BOOK VALUE OF EQUITY

N = 12,547 FIRM YEARS COVERING THE PERIOD FROM 1988 UNTIL 1998
FIGURE III

TWENTY ORDERED SUB-GROUP AVERAGES OF RATIO OF MARKET VALUE OF EQUITY TO BOOK VALUE OF EQUITY AGAINST RATIO OF EARNINGS TO BOOK VALUE OF EQUITY

N = 12,547 FIRM YEARS COVERING THE PERIOD FROM 1988 UNTIL 1998
PROOF OF THE AGGREGATION THEOREM

Part (ii) of the Aggregation Theorem assumes that the stochastic process which generates residual income is:

\[ da(t) = (c_{11}a(t) + c_{12}v(t))dt + k_1 \eta \delta(t)dz_1(t) \]

The expected present value of the stream of abnormal earnings is \( E_0[\int_0^\infty e^{-it}a(t)dt] \), where \( E_0(\cdot) \) is the expectations operator taken at time zero. Integrating by parts and invoking the transversality conditions:

(a) \( \lim_{t \to \infty} e^{-it}E_0[a(t)] = 0 \)

(b) \( \lim_{t \to \infty} e^{-it}E_0[v(t)] = 0 \)

implies:

\[ (1 - \frac{c_{11}}{i})E_0[\int_0^\infty e^{-it}a(t)dt] = \frac{a(0)}{i} + \frac{c_{12}}{i}E_0[\int_0^\infty e^{-it}v(t)dt] \]

Now, part (ii) of the Aggregation Theorem also assumes that the stochastic process which describes the evolution of the information variable is:

\[ dv(t) = (c_{21}a(t) + c_{22}v(t))dt + k_2 \eta \delta(t)dz_2(t) \]

We can use this assumption and similar procedures to those applied to the residual income variable to show that:

\[ (1 - \frac{c_{22}}{i})E_0[\int_0^\infty e^{-it}v(t)dt] = \frac{v(0)}{i} + \frac{c_{21}}{i}E_0[\int_0^\infty e^{-it}a(t)dt] \]

We then have two equations in two unknowns which solve to give:

\[ E_0[\int_0^\infty e^{-it}a(t)dt] = \frac{(i - c_{22})a(0)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} + \frac{c_{12}v(0)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} \]

From part (i) of the Aggregation Theorem the recursion value of equity will then be:
\[ \eta(t) = b(t) + E_t\left[\int_t^\infty e^{i(s-t)}a(s)ds\right] = b(t) + \frac{(i - c_{22})a(t)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} + \frac{c_{12}v(t)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} \]

Differentiating through this expression shows:

\[ d\eta(t) = db(t) + \frac{(i - c_{22})da(t)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} + \frac{c_{12}dv(t)}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} \]

Invoking the clean surplus identity, \( db(t) = (a(t) + ib(t))dt \), and substituting the expressions for \( da(t) \) and \( dv(t) \) into \( d\eta(t) \) gives:

\[ d\eta(t) = i\eta(t)dt + \eta_d(t)\delta(t) dq(t) \]

where \( dq(t) = (dz_1(t) + dz_2(t)) \) is a Wiener process with variance parameter \( \sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2 \).

Now, equity value satisfies the well known recursive relationship:

\[ P(\eta(t)) = e^{-idt}E_t[P(\eta(t + dt))] \]

where \( P(\eta(t)) \) is the expected present value of future dividends receivable from a unit investment in equity. If we expand \( P(\eta(t + dt)) \) as a Taylor series about the point \( \eta(t) \), use the fact that \( e^{-idt} = 1 - idt + \frac{1}{2}(idt)^2 + \ldots \), take expectations across the right hand side of the above recursive relationship, substitute the expressions for \( E_t[d\eta(t)] = i\eta(t)dt \) and \( \text{Var}_t[d\eta(t)] = (\sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2)\eta^2 dt \), divide both sides by \( dt \) and then take limits in such a way as to let \( dt \to 0 \), then the recursive relationship implies that equity value will also have to satisfy the canonical form of the fundamental valuation equation:14

\[ \frac{1}{2} \left[ \frac{\partial^2 P}{\partial a^2} + 2k_1 \frac{\partial^2 P}{\partial \sigma_{12}^2} a^2 + 2k_2 \frac{\partial^2 P}{\partial \sigma_{12}^2} \sigma_{12}^2 + (c_{11} + c_{12} \sigma_{12} \nu) \frac{\partial^2 P}{\partial a \partial \nu} + (c_{21} + c_{22} \sigma_{12} \nu) \frac{\partial^2 P}{\partial \nu^2} + (a + ib) \frac{\partial P}{\partial \nu} - iP = 0 \right] \]

14 It bears emphasising that there is no “sleight of hand” in determining the value of equity in terms of its recursion value, which is the “aggregation” variable on which our analysis is based. The alternative, and more complicated approach involves working directly in terms of the underlying variables using the recursion formula, \( P(b(t),a(t),v(t)) = e^{-idt}E_t[P(b(t + dt),a(t + dt),v(t + dt))] \). Taylor series expansions similar to those used in the text will then show that equity value will also have to satisfy the partial differential equation [Tippett (2000, pp. 1093-1095)]:
\[
\frac{1}{2}(\sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2)\eta^{-2\delta} \frac{d^2P}{d\eta^2} + i\eta \frac{dP}{d\eta} - iP(\eta) = 0
\]

Substitution shows \( P_1(\eta) = \eta \) to be a solution to this equation. Hence, by reduction of order [Boyce and DiPrima (1969, pp. 103-104)]:

\[
P_2(\eta) = \eta \int_\eta^\infty \frac{\exp\left[-\theta y^2(1-\delta)\right]}{\gamma^2} dy
\]

where \( \theta = \frac{2i}{\sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2} \), will be a second linearly independent solution of this differential equation. Note, however, that if we make the substitutions \( z = \frac{1}{\gamma} \) and \( dz = -\frac{dy}{\gamma^2} \) in this second solution, then it reduces to the simpler expression:

\[
P_2(\eta) = \eta \int_0^\frac{1}{\eta} \exp\left[-\theta z^2(\delta-1)\right] dz
\]

Furthermore, since \( P_1(\eta) \) captures the recursion value of equity it follows that \( P_2(\eta) \) must capture the adaptation value of equity. Finally, it is well known that every solution of the fundamental valuation equation can be expressed as a linear combination of the “fundamental” solutions \( P_1(\eta) \) and \( P_2(\eta) \) determined above [Boyce and DiPrima (1969, p. 93)]. This combined with the boundary conditions that equity value is exclusively made up of its adaptation value when \( \eta = 0 \) and exclusively made up of its recursion value as \( \eta \to \infty \), is sufficient to guarantee that:

\[
\eta = \frac{(i - c_{22})a}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}} + \frac{c_{12}v}{(i - c_{11})(i - c_{22}) - c_{21}c_{12}}
\]

is the recursion value of equity.

However, the substitution \( P(b,a,\nu) = P(\eta) \) reduces this equation to the simpler canonical form based on the ordinary differential equation used in the text. There are clear pedagogical and computational advantages in working with the canonical form based on recursion value rather than the more complicated partial differential equation from which it is derived.
\[
P(\eta) = \eta + P(0) \eta \int_{0}^{\frac{1}{\eta}} \exp\left[-\theta z^{2(\delta-1)} \frac{2(1-\delta)}{2(1-\delta)} \right] dz
\]
is the unique solution of the fundamental valuation equation.
REFERENCES


Burgstahler, D., “Discussion of ‘Combining Earnings and Book Value in Equity Valuation’”, Contemporary Accounting Research, 15, 3 (Fall 1998), pp. 325-341.


